

Positive feedback in coordination games: stochastic evolutionary dynamics and the logit choice rule

Sung-Ha Hwang^{a,*}, Luc Rey-Bellet^b

^a*College of Business, Korea Advanced Institute of Science and Technology (KAIST), Seoul, Korea*

^b*Department of Mathematics and Statistics, University of Massachusetts Amherst, MA, U.S.A.*

Abstract

We show that under the logit dynamics, positive feedback among agents (also called bandwagon property) induces evolutionary paths along which agents repeat the same actions consecutively so as to minimize the payoff loss incurred by the feedback effects. In particular, for paths escaping the domain of attraction of a given equilibrium—called a convention—positive feedback implies that along the minimum cost escaping paths, agents always switch first from the status quo convention strategy before switching from other strategies. In addition, the relative strengths of positive feedback effects imply that the same transitions occur repeatedly in the cost minimizing escape paths. By combining these two effects, we show that in an escaping transition from one convention to another, the least unlikely escape paths from the status quo convention consist of only the repeated identical mistakes of agents. Using our results on the exit problem, we then characterize the stochastically stable states under the logit choice rule for a class of non-potential games with an arbitrary number of strategies.

Keywords: Evolutionary Games, Logit Choice Rules, Positive Feedback, Bandwagon Property, Exit Problems, Stochastic Stability

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Email addresses: sungha.email@gmail.com (Sung-Ha Hwang), luc@math.umass.edu (Luc Rey-Bellet)

1. Introduction

We develop methods to find the most probable—or rather, least unlikely—paths of evolutionary dynamics with one of the most commonly used state-dependent mistake models, the logit choice rule. Evolutionary games and models are often applied to explain various durable social phenomena or individual characteristics that form over time and last in the long run. Examples include the evolution of social norms, convention, culture, languages, and preferences (see [Bowles \(2004\)](#) and [Young \(1998b\)](#))¹. Recently, one specific behavioral rule of myopic agents, the so-called logit choice model, has become popular among researchers because of its empirical plausibility as well as analytic convenience². Under the logit choice rule, the probability of agents’ mistakes decreases log-linearly in the payoff losses incurred by such mistakes; this is consistent with the economic rationale that agents are more cautious in avoiding mistakes that cause high payoff losses ([van Damme and Weibull, 2002](#)).

While modeling the emergence of social norms and conventions using binary choice or potential games—one of the popular modeling methods—provides useful insights for the problems at hand, explicitly accounting for more than two or three strategies and analyzing general non-potential games often yield rich and strong results. For example, [Young \(1998a\)](#) uses a discretized version of a continuous bargaining game (and hence a non-potential game with an arbitrary number of strategies) and proves that the bargaining norms of only minimally forward-looking agents with limited information approximate the Kalai–Smorodinsky solution—a highly rational and axiomatic cooperative bargaining solution (see also [Section 7](#)). Despite the recently prevailing use of the logit choice rule, no study in the literature shows how to analyze the rule for non-potential games with an *arbitrary* number of strategies. This

¹For example, [Young and Burke \(2001\)](#) study the evolution of contract norms between landowners and tenants. [Belloc and Bowles \(2013\)](#) examine the persistence and innovation of cultural-institutional conventions. [Alger and Weibull \(2013\)](#) study the evolution of preferences under incomplete information and show that the preferences called *homo moralis*—a mix of selfish and morality strands—are evolutionarily stable. [Lieberman, Michel, Jackson, Tang, and Nowak \(2007\)](#) study the question of quantifying the dynamics of language evolution.

²[Blume \(1993\)](#) introduced the logit choice rule to the economics literature. Recently, [Kreindler and Young \(2013\)](#), [Alós-Ferrer and Netzer \(2010\)](#), and [Okada and Tercieux \(2012\)](#) studied problems related to fast convergence, various revision rules, and local potentials under the logit choice rule. [Staudigl \(2012\)](#) and [Sandholm and Staudigl \(2016\)](#) study the problems of first exit and stochastic stability in a general setting and apply their results to a model under the logit choice rule. [Hwang, Lim, Neary, and Newton \(2016\)](#) study bargaining problems by combining intentional idiosyncratic plays and the logit choice rule. The previously cited studies of [Young and Burke \(2001\)](#) and [Belloc and Bowles \(2013\)](#) also use the logit choice model.

study aims to fill this gap by providing a new method to analyze evolutionary games under the logit choice rule and thus warranting a wide range of applicability of the rule.

We show that under the logit choice rule, positive feedback of agents (to coordinate) plays a key role: positive feedback of agents that play a coordination game induces evolutionary paths along which agents repeat the same actions consecutively to minimize the payoff losses caused by the feedback effects. Positive feedback, also referred to as bandwagon effects or strategic complementary interaction depending on the context, refers to the situation in which the payoff for choosing an action is an increasing function of the number of agents choosing the same action (Katz and Shapiro, 1985; Cooper, 1999).

In particular, we study the problems of exit from the domain of attraction of a given equilibrium—called a convention—and stochastic stability of the evolutionary dynamics of coordination games. These typically involve complicated optimization problems with a high-dimensional objective functional. For paths escaping a given convention in the exit problem, positive feedback implies that along the minimum cost escaping paths, agents always switch first from the status quo convention strategy before switching from other strategies. In addition, the relative strengths of positive feedback effects imply that the same transitions occur repeatedly in the cost minimizing escaping paths. By combining these two effects, we show that in an escaping transition from one convention to another, the least unlikely escaping paths from the status quo convention involve only the repeated identical mistakes of agents (Theorems 3.1 and 5.1).

Our exit problem results show that the path with the minimum exit cost involves the repeated transitions of agents from the status quo to some alternative convention. The transition costs from one convention to another—the costs needed to compute stochastic stable states—are, by definition, bounded below by the minimum exit cost. In addition, the path involving the repeated transitions from the status quo to another convention is itself a transition path. Thus, our exit problem results show that the minimum transition cost from the status quo convention, also called the “radius” of absorbing states, can be estimated by the minimum exit cost. It is also known that the minimum transition costs, under some sufficient conditions, are enough to determine a stochastically stable state (Young (1993); Kandori and Rob (1998); Binmore, Samuelson, and Young (2003); see Section 6). In this way, we obtain the results on stochastic stability (Theorem 6.1) from this result on the exit problem.

This paper makes the following novel contributions. First, our method and results apply to coordination game models with an arbitrary number of strategies, whether potential or non-potential, under the logit choice rule. Generally, evaluating the probability of transitions in state-dependent mistake models such as the logit choice rule is a difficult problem, because the unlikelihood of mistakes varies from state to state (see, e. g., [Bergin and Lipman \(1996\)](#)) and the number of escape paths grows rapidly in a number of strategies. Among the few results on these problems for the logit dynamic, [Sandholm and Staudigl \(2016\)](#) derive continuous optimal control problems by taking a zero error rate limit and then an infinite population limit (called the “double limit”) and solve the resulting control problems. This method, while providing systematic approaches to tackle the exit and stability problems, requires solving a (possibly challenging) Hamilton-Jacobi equation associated with the optimal control problem. The existing results are limited to either three-strategy one-population models (the cited paper) or two-strategy two-population models ([Staudigl, 2012](#)).

Our novel approach uses positive feedback property of underlying coordination games; we first identify and remove a significant number of irrelevant paths from the candidate solutions to the optimization problem. In this way, we avoid solving the complicated infinite-dimensional control problem, but instead obtain a much simpler problem in a finite dimensional Euclidean space that can be easily solved using elementary calculus. From this simplified problem, we determine the optimal paths for the evolutionary dynamic of coordination games with an arbitrary number of strategies. Although we also take the infinite population limit in our last step as in [Sandholm and Staudigl \(2016\)](#), the difference is that we reduce the set of cost minimizing discrete paths before taking the infinite population limit. To our knowledge, this is the first study to provide general results for the exit problem under the logit dynamics for non-potential coordination games with an arbitrary number of strategies. These results can be applied to study the stochastic stability problem under some conditions, as explained earlier.

The second contribution of this work is to derive comparison principles for paths using two effects related to positive feedback—(i) the existence of positive feedback and (ii) the relative strengths of different positive feedback effects. Thus, we single out the factors contributing to the likeliness of evolutionary paths, elucidating how the underlying interaction structures affect evolutionary paths. Our results can thus be applied to other mistake models with similar interaction structures. For example, some of our comparison principles (Propositions [3.3](#) and [5.2](#)) are still valid for exponential better-reply dynamics, which are often used to

model the behaviors of boundedly rational agents with limited cognitive ability (see the discussion in Section 9).³ Interestingly, the relative strength of the positive feedback effects mentioned in (ii) is measured by the well-known condition for the potential games by Hofbauer (1985) and Monderer and Shapley (1996) (see equation (8)).

Our third contribution is applying our results to the bargaining problem under the intentional logit dynamic (see Section 7 for precise definition) and obtaining a new bargaining norm that emerges and persists as a convention through the decentralized bargaining processes among myopic agents. This bargaining norm, compared to the existing bargaining solutions, shows how the individual cost of mistakes matters under the logit choice rule.

Our idea of studying the positive feedback effects associated with the transition path is related to the cycle decomposition of the state space in the literature of statistical mechanics (Schnakenberg, 1976). In the context of stochastic evolutionary games, choosing proper cycles and proving the comparison principle involving them are not easy tasks. However, we manage to solve this problem in this paper and obtain our main results.

This paper is organized as follows. Section 2 introduces the basic setup and discusses in some detail an example to illustrate our methods. We present our main results of the exit problem in Section 3. While Section 4 proves our main result, Section 5 presents our results of the exit problem for two-population games. Section 6 explains how to apply our exit problems to stochastic stability problems under known sufficient conditions. We then apply them to the bargaining problem in Section 7. Section 8 discusses how to study the stochastic stability problem directly. Finally, Section 9 concludes the paper.

2. Stochastic Evolutionary Dynamics: Setup and Example

2.1. Basic setup

Consider one population of n agents matched to play a symmetric coordination game with strategy set $S = \{1, 2, \dots, |S|\}$ and payoff matrix A . We consider a coordination game in which every strategy is a strict Nash equilibrium (see Condition A in Section 3). The

³See the better reply dynamic in Friedman and Mezzetti (2001); Dindoš and Mezzetti (2006); Josephson (2008)

population state is described as a vector of fractions of agents using each strategy; that is, the state of the population is $x \in \Delta^{(n)}$, where

$$\Delta^{(n)} := \{(x_1, \dots, x_{|S|}) \in \frac{1}{n}\mathbb{Z}^{|S|} : \sum_i x_i = 1, x_i \geq 0 \text{ for all } i\}.$$

The expected payoff to an agent choosing strategy i at population state x is given by $\pi(i, x) := \sum_{j \in S} A_{ij}x_j$.

We consider a discrete time strategy updating process, defined as follows. At each period, a randomly chosen agent selects a new strategy, and the new population state induced by the agent's switching from strategy i to j is denoted by $x^{i,j}$; that is, $x^{i,j} := x - \frac{1}{n}e_i + \frac{1}{n}e_j$, where e_i and e_j are the i -th and j -th elements of the standard basis for $\mathbb{R}^{|S|}$, respectively. We also denote by $x^{(i,j)(k,l)}$ the state induced by the agents' consecutive transitions, first from i to j and then from k to l . The conditional probability that an agent with strategy i chooses new strategy j given population state x is specified by the logit choice rule (Blume, 1993),

$$\text{Logit choice rule: } p_\beta(j|i, x) = \frac{\exp(\beta\pi(j, x))}{\sum_l \exp(\beta\pi(l, x))}, \quad (1)$$

where $\beta > 0$ is a positive parameter interpreted as the degree of rationality (or noise level). That is, as β increases to ∞ , equation (1) converges to the so-called best-response rule, whereas as β decreases to 0, equation (1) converges to a choice rule assigning equal probabilities to each strategy—namely, a pure randomization rule. When β is finite, the probability of choosing strategy j increases because the agent expects a higher payoff from strategy j than from other strategies.

The transition probability of the updating dynamics is

$$P_\beta(x, x^{i,j}) = x(i)p_\beta(j|i, x)$$

for $i \neq j$, where factor $x(i)$ accounts for the fact that at each period, one agent is chosen to revise her strategy. The measure of unlikelihood of a transition in stochastic evolutionary

game theory is called a cost, $c(x, y)$, between two states, x, y :

$$c(x, y) := \begin{cases} -\lim_{\beta \rightarrow \infty} \frac{\ln P_\beta(x, y)}{\beta} & \text{if } y = x^{i,j} \text{ for some } i, j, i \neq j \\ 0 & \text{if } y = x \\ \infty & \text{otherwise} \end{cases}$$

which becomes

$$c(x, x^{i,j}) = \max\{\pi(l, x) : l \in S\} - \pi(j, x) \quad (2)$$

under the logit choice rule (1). In equation (2), the first term $\max\{\pi(l, x) : l \in S\}$ is the payoff to agents playing the best response, say \bar{m} , in population state x , and the second term is the payoff to agents playing a new strategy, j . Thus, when a strategy-revising agent adopts the best response \bar{m} , the cost of such action is zero, whereas she adopts a sub-optimal strategy, the cost is the payoff loss due to choosing this strategy instead of the best response. By comparison, under the uniform mistake model, $c(x, x^{i,j}) = 1$ if $j \neq \bar{m}$ and $c(x, x^{i,j}) = 0$ if $j = \bar{m}$; this shows that the cost in the uniform mistake model is state independent.

A convention is defined to be the state in which everyone plays the same strategy that is a strict Nash equilibrium of A . Thus, x is a convention if $x = e_{\bar{m}}$ for some \bar{m} ; this is a strict Nash equilibrium (recall that e_i is the i -th element of the standard basis of $\mathbb{R}^{|S|}$). Thus, at convention \bar{m} , strategy \bar{m} is the mutual best response of agents. Fixing our attention on one such convention, we will call convention \bar{m} a status quo convention in the exit problem.

From the cost of transition between the states in equation (2), we define the cost of a path. Path γ is a sequence of states, $\gamma := (x_1, x_2, \dots, x_T)$, such that $x_{t+1} = (x_t)^{i,j}$ for some i, j and for all t . Then, the cost of path γ is defined as the sum of all costs of the intermediate transitions:

$$I(\gamma) := \sum_{t=1}^{T-1} c(x_t, x_{t+1}). \quad (3)$$

Using this setup, we present a simple example of a three-strategy game and illustrate the main ideas and results of the paper.

2.2. Illustration of the main results

Consider a technology choice game consisting of three technologies indexed by 1, 2, and 3 respectively. For example, as PC operating systems, one can have Windows, OSX, and Linux. Let b_i be the benefit that a technology i user obtains when interacting with another user of the same technology; thus, b_i is related to the inherent quality of technology i . Suppose that the user of technology i experiences some utility or disutility when interacting with users of a different technology. For simplicity, the users of technologies 2, 3, 1 derive utility d when interacting with the users of technologies 1, 2, 3, respectively. By the same token, the users of technologies 1, 2, 3 experience disutility d when interacting with the users of technologies 2, 3, 1, respectively. In sum, the payoff matrix is given by

$$A = \begin{pmatrix} b_1 & -d & d \\ d & b_2 & -d \\ -d & d & b_3 \end{pmatrix}. \quad (4)$$

In the context of the technology choice game, positive feedback means that the more the users of a given technology, the more the advantages of that technology. More precisely, the payoff advantage of technology i over j is greater when the other user adopts technology i instead of k :

$$A_{ii} - A_{ji} - (A_{ik} - A_{jk}) > 0, \quad (5)$$

for any distinct i, j, k . Condition (5) is called the marginal bandwagon property; it was introduced by [Kandori and Rob \(1998\)](#). If

$$3d < \min_i b_i \quad (6)$$

holds, all pure strategies 1, 2, and 3 are strict Nash equilibria and condition (5) is satisfied. For example, if we take $i = 1$ and $j, k = 2, 3$, respectively, then condition (5) becomes

$$A_{11} - A_{31} - (A_{12} - A_{32}) = b_1 + 3d > 0, \quad A_{11} - A_{21} - (A_{13} - A_{23}) = b_1 - 3d > 0. \quad (7)$$

We can also compare the degree (or extent) of the first positive feedback effect with that of

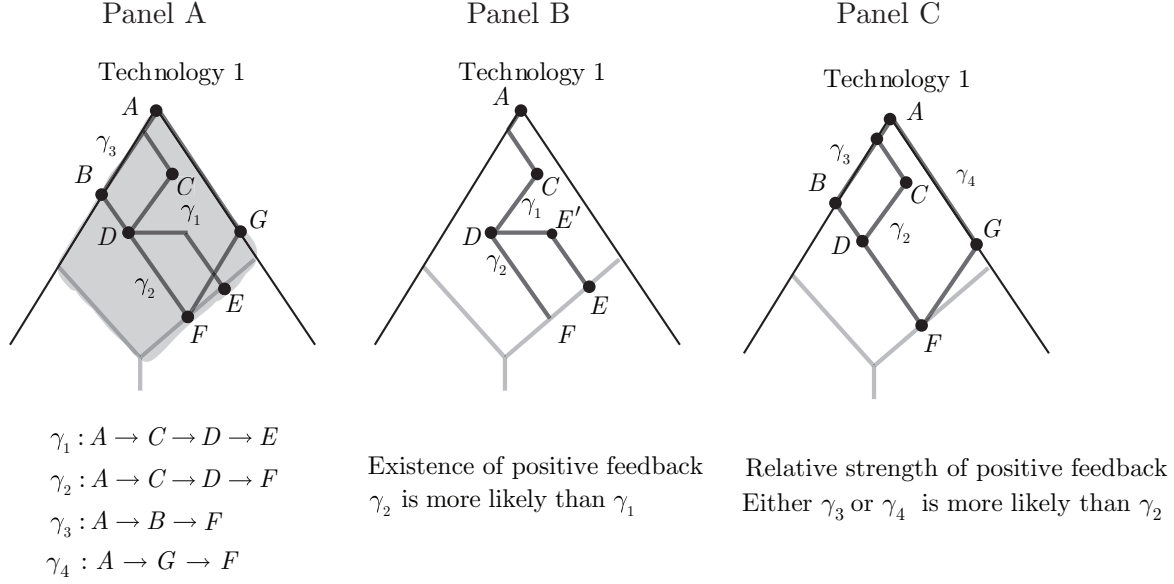


Figure 1: **Basin of attraction, first exit problems, and comparison of paths**

the second one in equation (7) as follows:

$$[A_{11} - A_{31} - (A_{12} - A_{32})] - [A_{11} - A_{21} - (A_{13} - A_{23})] = A_{21} - A_{12} + A_{13} - A_{31} + A_{32} - A_{23} = 6d. \quad (8)$$

Here, $6d$ in equation (8) can be positive or negative (or zero). When $6d = 0$, the game is a potential game and this is a well-known test for potential games by [Hofbauer \(1985\)](#) and [Monderer and Shapley \(1996\)](#).

Suppose that the agents' strategy revision rule is the logit choice rule. Our example is the problem of the exit from a convention (see [Freidlin and Wentzell \(1998\)](#)). Specifically, given the status quo convention of technology 1, what is the least unlikely way to upset this convention? The shaded region in Panel A of Figure 1 shows the basin of attraction of technology 1 convention, $D(e_1)$:

$$D(e_1) := \{x \in \Delta^{(n)} : \pi(1, x) \geq \pi(k, x) \text{ for all } k\}$$

from which the population dynamic has to escape via the agents' non-best response plays.

Our problem can thus be succinctly stated as

$$\min\{I(\gamma) : \gamma \text{ escapes } D(e_1)\}. \quad (9)$$

We now explain how the two conditions in (7) and (8) significantly reduce the complexity of solving the minimization problem in (9).

Panel A in Figure 1 depicts four escaping paths for the stochastic evolutionary dynamic. If the agents' technology choice rules follow the uniform mistake model, the least unlikely path can be easily identified by counting the number of mistakes in the paths. Since in the simplex of the state space, the number of transitions is represented as the length of the corresponding path, the shortest escaping path is the least unlikely escaping path under the uniform mistake rule. However, when the mistake model is given by the logit choice rule, this simple comparison is impossible. This is because the unlikeliness of a path under the logit choice rule depends on an individual's opportunity cost for each mistake in the path as well as for the total number of mistakes involved in it. Thus, *a priori*, one cannot easily find the minimum cost path for escaping the basin of attraction.

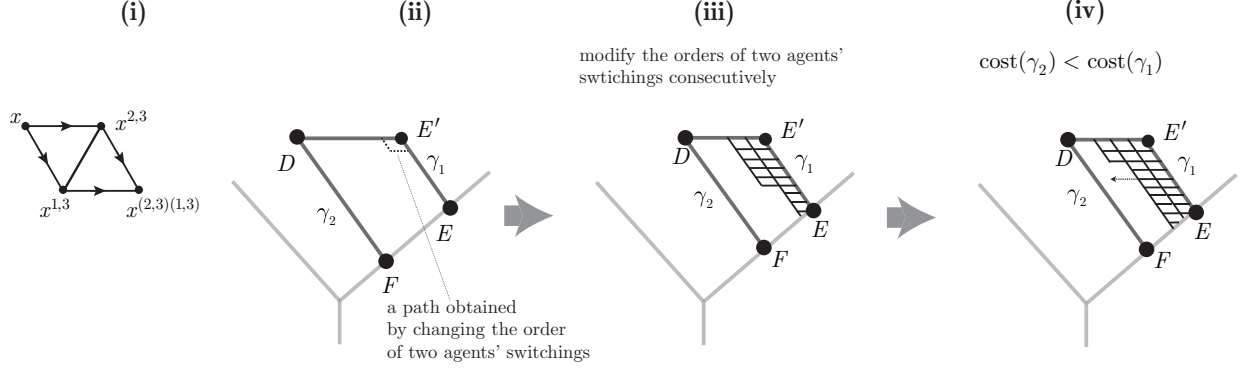
Comparison principle 1: Propositions 3.1 and 4.1, Part (i)

Our idea is to develop systematic ways to compare the costs of various paths and reduce the number of candidate solutions to the minimization problem in (9). To explain this in more detail, consider the four paths depicted in Figure 1: γ_1 , γ_2 , γ_3 , and γ_4 . We first show how the existence of positive feedback (or bandwagon effects) implies that the cost of γ_2 is cheaper than that of γ_1 . Note that to compare γ_1 with γ_2 , it is enough to compare the path from D through E' to E and the path from D to F in Panel B of Figure 1. To compare these two paths, we develop the first comparison principle as follows. First, consider the two paths in (i) of Panel A in Figure 2: (1) $x \rightarrow x^{2,3} \rightarrow x^{(2,3)(1,3)}$ and (2) $x \rightarrow x^{1,3} \rightarrow x^{(1,3)(2,3)}$, where $x^{(2,3)(1,3)} = x^{(1,3)(2,3)}$. The difference between the costs of these two paths is

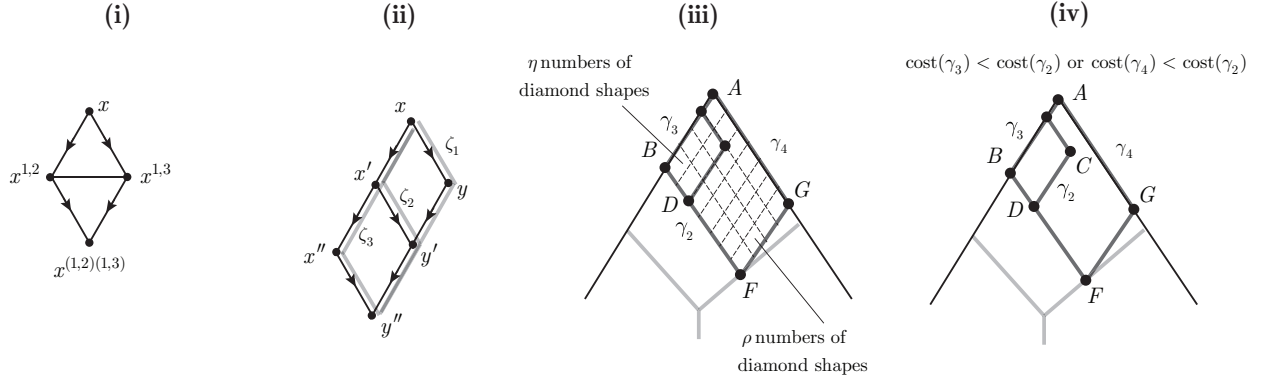
$$\Delta_1 c := c(x, x^{2,3}) + c(x^{2,3}, x^{(2,3)(1,3)}) - [c(x, x^{1,3}) + c(x^{1,3}, x^{(1,3)(2,3)})]. \quad (10)$$

Note that under the logit choice rule an agent switching from strategy 2 to strategy 3 compares the expected payoff of strategy 3 with the best response (strategy 1) rather than with strategy 2, implying that the costs of transitions from strategy 2 to strategy 3 and from

Panel A: Existence of positive feedback (Comparison Principle 1: Propositions 3.1, 4.1 (i))



Panel B: Relative strength of positive feedback effects (Comparison Principle 2: Propositions 3.2, 3.3, 4.1 (ii))



Panel C: Remaining candidate paths (Propositions 4.3, 4.4)

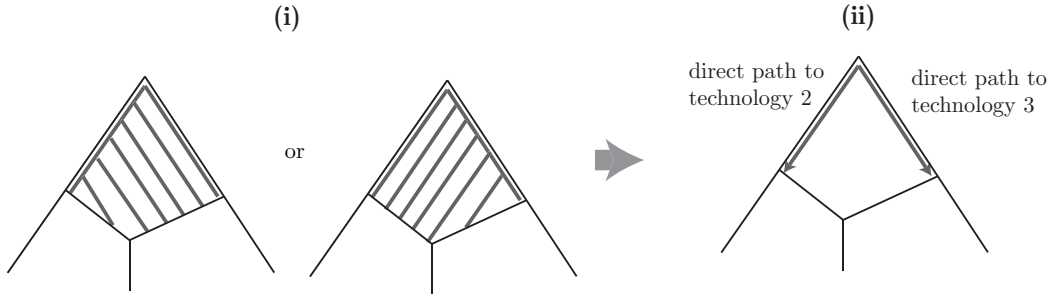


Figure 2: **Comparison Principles.** Panels A and B show how the positive feedback effects and the associated relative strength of these effects yield paths with lower costs.

strategy 1 to strategy 3 are the same. Indeed from equation (2) we verify that

$$c(x, x^{2,3}) = c(x, x^{1,3}), \quad c(x^{1,3}, x^{(1,3),(2,3)}) = c(x^{1,3}, x^{(1,3),(1,3)}) \quad (11)$$

Thus, from equation (11), we can simplify (10) as follows:

$$\Delta_1 c = c(x^{2,3}, x^{(2,3)(1,3)}) - c(x^{1,3}, x^{(1,3),(1,3)})$$

Note that $x^{2,3}$ is the state where exactly one more agent than in $x^{1,3}$ plays strategy 1, since $x^{2,3} - x^{1,3} = \frac{1}{n}(e_1 - e_2)$. The positive feedback effect of strategy 1 over strategy 3 means that the payoff advantage of strategy 1 over strategy 3 is greater when the other player uses strategy 1. Thus, this advantage is greater at the state $(x^{2,3})$ where one more agent than in the other state $(x^{1,3})$ plays strategy 1. This means that the payoff loss due to the mistake of playing strategy 3 (rather than strategy 1) is greater at $x^{2,3}$ than at $x^{1,3}$. We thus expect that under the logit dynamic, the transition from strategy 1 to strategy 3 will be more costly at $x^{2,3}$ than at $x^{1,3}$. Indeed, we find that

$$\begin{aligned} \Delta_1 c &= c(x^{2,3}, x^{(2,3)(1,3)}) - c(x^{1,3}, x^{(1,3)(1,3)}) = \pi(1, x^{2,3}) - \pi(3, x^{2,3}) - [\pi(1, x^{1,3}) - \pi(3, x^{1,3})] \\ &= \frac{1}{n}(A_{11} - A_{31} - (A_{12} - A_{32})) > 0 \end{aligned} \quad (12)$$

which is positive from condition (7). Equation (12) also shows that when agents make the same mistake (switching from 1 to 3), the cost becomes cheaper ($c(x^{1,3}, x^{(1,3)(1,3)}) < c(x^{2,3}, x^{(2,3)(1,3)})$). Thus under the assumption of positive feedback, the cost of path $x \rightarrow x^{1,3} \rightarrow x^{(1,3)(2,3)}$ is cheaper than that of $x \rightarrow x^{2,3} \rightarrow x^{(2,3)(1,3)}$ (see Proposition 3.1).

Now, consider the new path (shown as a dotted line) obtained by altering a single agent's switching in (ii) of Panel A of Figure 2. The cost difference between the original and new paths in (ii) of Panel A of Figure 2 is precisely the cost difference between the two paths in (i) of the panel. Thus, if equation (12) is positive, the cost of the new path in (ii) of Panel A is strictly lower than that of the original path. Then, by successively altering a single agent's switching, we can apply the same arguments repeatedly as in (iii) and (iv) of Panel A of Figure 2 (Proposition 4.1, Part (i)). In addition, because $c(x, x^{1,3}) = c(x, x^{2,3})$ in equation (11), we find that the cost of $x \rightarrow x^{1,3}$ is equal to or less than the costs of $x \rightarrow x^{2,3}$ and $x \rightarrow x^{2,3} \rightarrow x^{(2,3)(1,3)}$. Thus, the above modification argument holds irrespective of whether the path lies in the boundary or interior of the basin of attraction. In this way we find that

the cost of path $D \rightarrow F$ is cheaper than that of path $D \rightarrow E' \rightarrow E$ and the cost of path γ_2 is smaller than that of path γ_1 .

Comparison principle 2: Propositions 3.3, 3.2, and 4.1, Part (ii)

Next, we compare three paths, γ_2 , γ_3 , and γ_4 in Panel C of Figure 1. Similarly, we first consider two paths in (i) of Panel B of Figure 2: (1) $x \rightarrow x^{1,3} \rightarrow x^{(1,3)(1,2)}$ and (2) $x \rightarrow x^{1,2} \rightarrow x^{(1,2)(1,3)}$, where $x^{(1,3)(1,2)} = x^{(1,2)(1,3)}$.

$$\begin{aligned} \Delta_2 c := & c(x, x^{1,3}) + c(x^{1,3}, x^{(1,3)(1,2)}) - [c(x, x^{1,2}) + c(x^{1,2}, x^{(1,2)(1,3)})] \\ = & \underbrace{[c(x, x^{1,3}) - c(x^{1,2}, x^{(1,2)(1,3)})]}_{\text{(i) positive feedback of 1 over 3}} - \underbrace{[c(x, x^{1,2}) - c(x^{1,3}, x^{(1,3)(1,2)})]}_{\text{(ii) positive feedback of 1 over 2}} \end{aligned} \quad (13)$$

If we let $x = y^{2,3}$, then $x^{1,2} = y^{1,3}$ and, as in equation (12), (i) in equation (13) becomes

$$\begin{aligned} c(x, x^{1,3}) - c(x^{1,2}, x^{(1,2)(1,3)}) &= c(y^{2,3}, y^{(2,3)(1,3)}) - c(y^{1,3}, y^{(1,3)(1,3)}) \\ &= \frac{1}{n}(A_{11} - A_{31} - (A_{12} - A_{32})). \end{aligned}$$

Furthermore, if we let $x = z^{3,2}$, then $x^{1,3} = z^{1,2}$ and (ii) in equation (13) becomes

$$\begin{aligned} c(x, x^{1,2}) - c(x^{1,3}, x^{(1,3)(1,2)}) &= c(z^{3,2}, z^{(3,2)(1,2)}) - c(z^{1,2}, z^{(1,2)(1,2)}) \\ &= \frac{1}{n}(A_{11} - A_{21} - (A_{13} - A_{23})), \end{aligned}$$

the positive feedback effect of strategy 1 over 2. We then verify that

$$\Delta_2 c = \frac{1}{n}(-A_{12} + A_{13} + A_{21} - A_{23} - A_{31} + A_{32}) = 6\frac{d}{n}. \quad (14)$$

Next, using this, we compare three paths in Panel B (ii) of Figure 2, defined as follows:

$$\zeta_1 : x \rightarrow y \rightarrow y' \rightarrow y'' \quad (15)$$

$$\zeta_2 : x \rightarrow x' \rightarrow y' \rightarrow y'' \quad (16)$$

$$\zeta_3 : x \rightarrow x' \rightarrow x'' \rightarrow y'' \quad (17)$$

Then, applying equation (14), we find that

$$I(\zeta_1) - I(\zeta_2) = 6d \text{ and } I(\zeta_2) - I(\zeta_3) = 6d \quad (18)$$

which show that

$$[I(\zeta_2) - I(\zeta_1)] + [I(\zeta_2) - I(\zeta_3)] = 0. \quad (19)$$

This implies that either

$$I(\zeta_2) > I(\zeta_1), I(\zeta_2) > I(\zeta_3) \text{ or } I(\zeta_1) = I(\zeta_2) = I(\zeta_3) \quad (20)$$

holds. Thus, from equation (20) either ζ_1 or ζ_3 costs less than (or equal to) ζ_2 and in this way, we can remove ζ_2 from the candidate paths minimizing the problem in equation (9).

Next, we consider three paths, γ_2, γ_3 , and γ_4 in Panel C of Figure 2. Recall that

$$\gamma_2 : A \rightarrow C \rightarrow D \rightarrow F \quad (21)$$

$$\gamma_3 : A \rightarrow B \rightarrow F \quad (22)$$

$$\gamma_4 : A \rightarrow G \rightarrow F \quad (23)$$

First, consider the area between γ_4 and γ_2 and assume that there are ρ diamond shapes in this area (see Panel B (iii) of Figure 2). Now, by applying the comparison result of $I(\zeta_1) - I(\zeta_2)$ in equation (18), we find that

$$I(\gamma_4) - I(\gamma_3) = \rho 6d. \quad (24)$$

Similarly, assume that there are η diamond shapes between γ_2 and γ_3 . By applying the comparison result in equation (18), we find that

$$I(\gamma_2) - I(\gamma_3) = \eta 6d. \quad (25)$$

From equations (24) and (25), we find that

$$\eta[I(\gamma_2) - I(\gamma_4)] + \rho[I(\gamma_2) - I(\gamma_3)] = 0 \quad (26)$$

which in turn implies that either

$$I(\gamma_2) > I(\gamma_4), I(\gamma_2) > I(\gamma_3) \text{ or } I(\gamma_2) = I(\gamma_4) = I(\gamma_3) \quad (27)$$

holds. Using this, we can also remove γ_2 from the minimum cost candidate paths (see Panel B (iv) of Figure 2).

Finally, we apply these two comparison principles, to obtain a class of paths comprising the candidate solutions to the cost minimization problem in equation (9) (see Panel C (i) of Figure 2). These paths consist of consecutive transitions first from technology 1 to technology 2 and then from technology 1 to technology 3 or, alternatively, consecutive transitions first from technology 1 to technology 3 and then from technology 1 to technology 2. We thus reduce the complicated objective function in equation (9) to a function of two variables (i.e., number of transitions from technology 1 to technology 2 and from technology 1 to technology 3) and easily study the minimization problem of this simple objective function relying on elementary calculus. We then prove that the lowest cost transition path to escape the basin of attraction of convention 1 involves the repetition of the same kind of mistakes. That is, graphically, these paths lie on the edges of the simplex from strategy 1 to strategy 2 and from strategy 1 to strategy 3 (see Panel C (ii) of Figure 2 and Propositions 4.3 and 4.4). In general, we can reduce the objective function with an arbitrary number of variables in equation (9) to an objective function with $|S| - 1$ variables, where $|S|$ is the number of strategies of the underlying game.

3. Exit from a basin of attraction of a convention : one-population models

In this section, we explain the key steps of our approach and state our main result on escaping from the basin of attraction of a convention. First, recall that the marginal bandwagon properties introduced by Kandori and Rob (1998) represent the positive feedback effects in coordination games. A symmetric game with payoff matrix A satisfies *the marginal bandwagon property* (**MBP**): if

$$A_{ii} - A_{ji} > A_{ik} - A_{jk} \text{ for all distinct } i, j, k. \quad (28)$$

We also consider a coordination game in which $A_{ii} > A_{ji}$ for all i, j and assume that all kinds of mixed-strategy Nash equilibria exist; that is, for any arbitrary subset of S there exists a mixed-strategy Nash equilibrium whose support is that subset. Thus, we consider the following class of games.

Condition A: Suppose that the one-population game with payoff matrix A is a coordination game (i.e., $A_{ii} > A_{ji}$ for all i, j) and satisfies **MBP**, and that for any $T \subset S$, there exists a mixed-strategy Nash equilibrium with support T .

We define the basin of attraction of $e_{\bar{m}}$, $D(e_{\bar{m}})$, and the (exterior) boundary, $\partial D(e_{\bar{m}})$, respectively, as follows:

$$\begin{aligned} D(e_{\bar{m}}) &:= \{x \in \Delta^{(n)} : \pi(\bar{m}, x) \geq \pi(k, x) \text{ for all } k\}, \\ \partial D(e_{\bar{m}}) &:= \{y \notin D(e_{\bar{m}}) : P_\beta(x, y) > 0 \text{ for some } x \in D(e_{\bar{m}})\}. \end{aligned}$$

In fact, if x belongs to $D(e_{\bar{m}})$, the cost of a transition from i to j is

$$c(x, x^{i,j}) = \pi(\bar{m}, x) - \pi(j, x) \quad (29)$$

for $i \neq j$. If $j = \bar{m}$, the cost in equation (29) is zero and **MBP** implies that convention \bar{m} can be reached from any $x \in D(e_{\bar{m}})$ at no cost; thus, $D(e_{\bar{m}})$ is indeed the basin of attraction of convention \bar{m} .

As regards to the cost function c , an immediate consequence of **MBP** is that given two paths $(x \rightarrow x^{(i,k)} \rightarrow x^{(i,k)(\bar{m},l)})$ and $(x \rightarrow x^{(\bar{m},k)} \rightarrow x^{(i,k)(\bar{m},l)})$ in Panel A of Figure 3, it always costs less (or the same) to first switch away from strategy \bar{m} and then switch away from the other strategies, as explained in Section 2. This plays a central role in determining the optimal escape paths in the sequel.

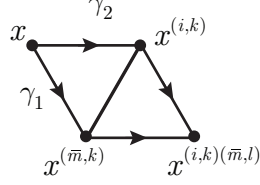
Proposition 3.1 (Comparison Principle 1). *Suppose that **MBP** holds. Consider two paths γ_1 and γ_2 (Panel A of Figure 3) in $D(e_{\bar{m}})$:*

$$\begin{aligned} \gamma_1 &: x \rightarrow x^{\bar{m},k} \rightarrow x^{(\bar{m},k)(i,l)}, \\ \gamma_2 &: x \rightarrow x^{i,k} \rightarrow x^{(i,k)(\bar{m},l)}, \end{aligned}$$

where $i \neq \bar{m}, k \neq i, \bar{m}, l \neq i, \bar{m}$. Then,

$$I(\gamma_1) < I(\gamma_2).$$

Panel A: Existence of positive feedback effects



Panel B: Relative strength of positive feedback effects

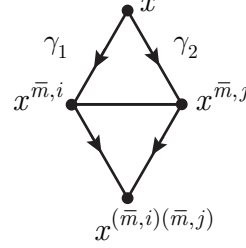


Figure 3: **The existence and relative strength of positive feedback effects**

Proof. Since $c(x, x^{i,k}) = \pi(\bar{m}, x) - \pi(k, x)$, we obtain

$$\begin{aligned}
 I(\gamma_2) - I(\gamma_1) &= [\pi(\bar{m}, x) - \pi(k, x) + \pi(\bar{m}, x^{i,k}) - \pi(l, x^{i,k})] \\
 &\quad - [\pi(\bar{m}, x) - \pi(k, x) + \pi(\bar{m}, x^{\bar{m},k}) - \pi(l, x^{\bar{m},k})] \\
 &= \frac{1}{n} ([-A_{\bar{m}i} + A_{\bar{m}k} + A_{li} - A_{lk}] - [-A_{\bar{m}\bar{m}} + A_{\bar{m}k} + A_{l\bar{m}} - A_{lk}]) \\
 &= \frac{1}{n} (A_{\bar{m}\bar{m}} - A_{l\bar{m}} - A_{\bar{m}i} + A_{li}) > 0
 \end{aligned}$$

from **MBP**. □

From **MBP**, the degree of positive feedback effects of strategy i over j when the other player's alternative strategy is k can be measured by the following quantity:

$$H_{jk}^{(i)} := A_{ii} - A_{ji} - A_{ik} + A_{jk}. \quad (30)$$

If **MBP** is satisfied, $H_{jk}^{(i)}$ is non-negative for all distinct i, j, k . Note that the quantities $H_{jk}^{(i)}$ in equation (30) are in general not symmetric when exchanging j and k ; this asymmetry is related to the well-known criterion for potential games for a symmetric game with payoff matrix A (Hofbauer, 1985; Monderer and Shapley, 1996). This criterion states that a game is a potential game if and only if

$$B_{jk}^{(i)} := [A_{ji} - A_{ij}] + [A_{kj} - A_{jk}] + [A_{ik} - A_{ki}] = 0 \quad (31)$$

for all i, j, k . This quantity $B_{jk}^{(i)}$ is also called “skew” in Sandholm and Staudigl (2016). From equations (30) and (31), we can straightforwardly confirm that

$$H_{jk}^{(i)} - H_{kj}^{(i)} = -B_{jk}^{(i)}. \quad (32)$$

That is, the game is a potential game if and only if $H_{jk}^{(i)} = H_{kj}^{(i)}$.

From the following lemma, the difference in costs between two paths transiting away from convention \bar{m} (γ_1 and γ_2 in Panel B of Figure 3) is precisely the relative strengths of positive feedback effects.

Proposition 3.2 (Relative strengths of positive feedback effects). *Consider two paths γ_1 and γ_2 (Panel B of Figure 3) in $D(e_{\bar{m}})$:*

$$\begin{aligned} \gamma_1 : x &\rightarrow x^{\bar{m},i} \rightarrow x^{(\bar{m},i)(\bar{m},j)}, \\ \gamma_2 : x &\rightarrow x^{\bar{m},j} \rightarrow x^{(\bar{m},j)(\bar{m},i)}, \end{aligned}$$

where $i \neq j$. Then, we have

$$I(\gamma_2) - I(\gamma_1) = \frac{1}{n} B_{ij}^{(\bar{m})}$$

Proof. We have

$$\begin{aligned} I(\gamma_2) - I(\gamma_1) &= [\pi(\bar{m}, x) - \pi(j, x) + \pi(\bar{m}, x^{\bar{m},j}) - \pi(i, x^{\bar{m},j})] \\ &\quad - [\pi(\bar{m}, x) - \pi(i, x) + \pi(\bar{m}, x^{\bar{m},i}) - \pi(j, x^{\bar{m},i})] \\ &= \frac{1}{n} (A_{ji} - A_{j\bar{m}} + A_{\bar{m}j} - A_{\bar{m}i} - A_{ij} + A_{i\bar{m}}) \end{aligned}$$

□

The important consequence of Proposition 3.2 is that the cost difference between two paths, γ_1 and γ_2 , depends on only the kind of strategies, not on the state, x . As explained in the subsection of Comparison principle 2 in Section 2, we can collect the same kind of transitions (that is, the transitions from strategy \bar{m} to the same strategy) using Lemma 3.2. To state this fact more precisely, we denote by $(\bar{m}, k; \eta)$ η -times consecutive transitions from \bar{m} to k . Also, let $x^{\bar{m},k,\eta}$ be a new state induced by the agents’ η -times consecutive switches from \bar{m} to k from an old state, x .

Proposition 3.3 (Comparison Principle 2). *Consider the following paths (see Panel B of Figure 2):*

$$\begin{array}{llllllll}
\gamma & : & x & \xrightarrow{(\bar{m}, k; \eta)} & x^{\bar{m}, k, \eta} & \longrightarrow & y & \cdots & z & \xrightarrow{(\bar{m}, k; \rho)} & z^{\bar{m}, k, \rho} \\
\gamma' & : & x & \xrightarrow{(\bar{m}, k; \eta)} & x^{\bar{m}, k, \eta} & \xrightarrow{(\bar{m}, k; \rho)} & x^{(\bar{m}, k, \eta)(\bar{m}, k, \rho)} & \longrightarrow & y^{\bar{m}, k, \rho} & \cdots & z^{\bar{m}, k, \rho} \\
\gamma'' & : & x & \longrightarrow & y^{k, \bar{m}, \eta} & \cdots & z^{k, \bar{m}, \eta} & \xrightarrow{(\bar{m}, k; \eta)} & z & \xrightarrow{(\bar{m}, k; \rho)} & z^{\bar{m}, k, \rho}
\end{array}$$

where \cdots denotes the same transitions. Then the following holds:

$$\eta[I(\gamma') - I(\gamma)] + \rho[I(\gamma'') - I(\gamma)] = 0.$$

Thus, either

$$I(\gamma') > I(\gamma) > I(\gamma''), I(\gamma'') > I(\gamma) > I(\gamma'), \text{ or } I(\gamma'') = I(\gamma) = I(\gamma')$$

holds.

Proof. See Appendix B. □

Next, we explain our main result on the exit problem from convention \bar{m} . We denote by $I^{(n)}(\gamma)$ the cost of path γ for a population of size n ; let $\mathcal{G}_{\bar{m}}^{(n)}$ be the set of all paths escaping the basin of attraction of convention \bar{m} at some arbitrary time T ; that is,

$$\mathcal{G}_{\bar{m}}^{(n)} := \{\gamma = (x_0, \cdots, x_T) : x_0 = e_{\bar{m}}, x_t \in D(e_{\bar{m}}) \text{ for } 0 < t < T - 1 \text{ and } x_T \notin D(e_{\bar{m}}) \text{ for some } T > 0\}. \quad (33)$$

Our optimization problem can thus be succinctly written as

$$\min\{I^{(n)}(\gamma) : \gamma \in \mathcal{G}_{\bar{m}}^{(n)}\}. \quad (34)$$

Using Propositions 3.1 and 3.3, we reduce significantly the number of candidate solutions to the problem in (34).

Even after eliminating a substantial number of irrelevant paths from the set of candidate solutions to the problem in (34), function $I^{(n)}$ still remains complicated, with negligible terms (in the order of n) when the population is large. Thus, we consider an infinite pop-

ulation limit; this provides an asymptotic characterization of the exit problem when n is large (see the discussion in Section 3.3 in [Binmore, Samuelson, and Young \(2003\)](#); see also [Sandholm and Staudigl \(2016\)](#) and [Hwang and Newton \(2016\)](#)). We define

$$R_{\bar{m}j} := \frac{1}{2} \frac{(A_{\bar{m}\bar{m}} - A_{j\bar{m}})^2}{(A_{\bar{m}\bar{m}} - A_{j\bar{m}}) + (A_{jj} - A_{\bar{m}j})}$$

and $nR_{\bar{m}j}$ is nothing but an approximate cost of the path along the edge of the simplex for a large n and this path is given by

$$\{x \in \Delta^{(n)} : x = e_{\bar{m}} + \frac{k}{n}(e_j - e_{\bar{m}}), 0 \leq k \leq n\}$$

which exits $D(e_{\bar{m}})$ at $q_{\{\bar{m}, j\}}$, the Nash equilibrium with support $\{\bar{m}, j\}$.

Theorem 3.1 (Exit problem). *Assume that Condition **A** holds. Then, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min_{\gamma \in \mathcal{G}_{\bar{m}}^{(n)}} I^{(n)}(\gamma) = R_{\bar{m}j^*}$$

where j^* satisfies

$$j^* = \arg \min \{R_{\bar{m}j} : j \neq \bar{m}\}. \quad (35)$$

Proof. See Section 4. □

In the much-studied uniform mistake model, the probabilities of mistakes are identical for all states and hence are state independent (see, e.g., [Binmore, Samuelson, and Young \(2003\)](#)). Therefore, the threshold number of the deviant agents inducing other agents to change their best responses is the only determinant of the expected escaping time and stochastic stability. The number $n \frac{(A_{\bar{m}\bar{m}} - A_{i\bar{m}})}{(A_{\bar{m}\bar{m}} - A_{i\bar{m}}) + (A_{ii} - A_{\bar{m}i})}$ is the threshold number of agents deviating from strategy \bar{m} to strategy i and inducing others to best respond with strategy i . It is known that under the uniform mistake model, when **MBP** holds,

$$\min_{\gamma \in \mathcal{G}_{\bar{m}}^{(n)}} I^{(n)}(\gamma) \approx \min \left\{ n \frac{(A_{\bar{m}\bar{m}} - A_{i\bar{m}})}{(A_{\bar{m}\bar{m}} - A_{i\bar{m}}) + (A_{ii} - A_{\bar{m}i})} : i \neq \bar{m} \right\}$$

indicating that only these threshold numbers matter in upsetting convention \bar{m} (see [Binmore, Samuelson, and Young \(2003\)](#); [Kandori and Rob \(1998\)](#)).

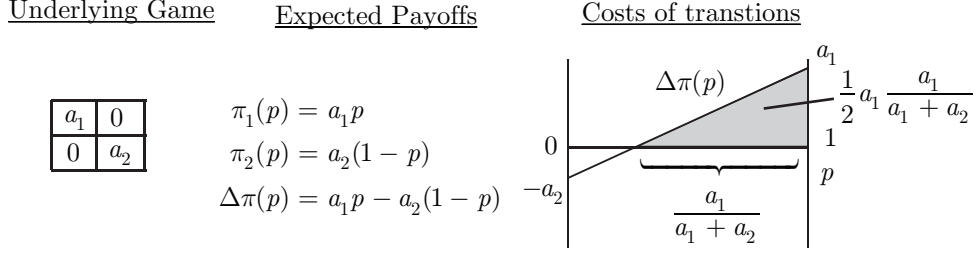


Figure 4: **Costs of uniform mistake and logit choice models.** Given the underlying game, let p be the population fraction using strategy 1. Then, the expected payoffs to strategies 1 and 2 are $\pi_1(p)$ and $\pi_2(p)$, respectively. The difference between $\pi_1(p)$ and $\pi_2(p)$, denoted by $\Delta\pi(p)$, can be regarded as the evolutionary selection force for strategy 1. Under the uniform mistake model, the cost of escaping the equilibrium of strategy 1 is simply the size of the basin of attraction of strategy 1 equilibrium, $\frac{a_1}{a_1+a_2}$. This cost under the logit choice rule is now the area of the shaded region, $\frac{1}{2} \times \frac{a_1}{a_1+a_2} \times a_1$. This cost thus accounts for the individual opportunity cost of miscoordination (e.g., when $p = 1$, this cost is a_1) as well as the threshold number of agents inducing others to adopt a different best response.

While this may be plausible in some situations, evolutionary selection may depend on also the individuals' opportunity costs of miscoordination as well as the threshold numbers. For the example in Figure 4, the uniform mistake model predicts the same (asymptotic) expected waiting time as long as a_1/a_2 remains the same; for example, $a_1 = 2, a_2 = 1$ versus $a_1 = 2000, a_2 = 1000$. However, if the agents are more cautious in avoiding mistakes at the equilibrium of strategy 1, it would take much longer to escape from the equilibrium of $a_1 = 2000$ than from the equilibrium of $a_1 = 2$. The logit choice model incorporates this (plausible) opportunity cost effect of miscoordination⁴. Thus, the evolutionary force in this model depends on both how “deep” and how “large” the basins of attraction of the conventions are (see Figure 4 and the introduction in [van Damme and Weibull \(2002\)](#)).

Thus, the cost depends on these two factors, with the “area” of the triangle in Figure 4 representing the evolutionary selection force for convention \bar{m} relative to convention i : $\frac{1}{2}n \frac{(A_{\bar{m}\bar{m}} - A_{i\bar{m}})^2}{(A_{\bar{m}\bar{m}} - A_{i\bar{m}}) + (A_{ii} - A_{mi})}$. From Theorem 3.1, the expected time of first exit from convention

⁴Indeed, we can compare the expected exit times under two models by using equation (36) as follows. Under the uniform- mistake and logit choice models, the mistake probabilities are typically parameterized by ϵ and β , where $\epsilon \approx 0$ and $\beta \approx \infty$ and the expected exit times are approximately given by $\epsilon^{-na_1/(a_1+a_2)}$ and $e^{n\beta/2} a_1^{1/(a_1+a_2)}$, respectively, where n is the total number of agents. We denote by τ_U and τ'_U the expected exit times under the uniform mistake model for $a_1 = 2, a_2 = 1$ and $a'_1 = 2000, a'_2 = 1000$, respectively and denote by τ_L and τ'_L the corresponding times under the logit choice rule. Then, we find that $\tau'_U = \tau_U$, but $\tau'_L = (\tau_L)^{1000}$.

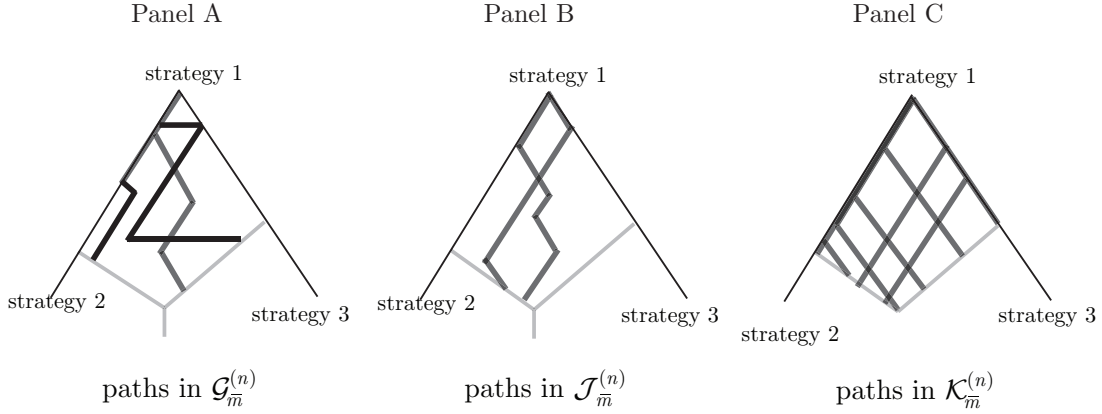


Figure 5: **Illustration of Proposition 4.1** Panels A, B, and C show the paths in $\mathcal{G}_{\bar{m}}^{(n)}$ in equation (33), $\mathcal{J}_{\bar{m}}^{(n)}$ in equation (37), and $\mathcal{K}_{\bar{m}}^{(n)}$ in equation (39), respectively. Part (i) of Proposition 4.1 reduces the set of candidate paths from $\mathcal{G}_{\bar{m}}^{(n)}$ (Panel A) to $\mathcal{J}_{\bar{m}}^{(n)}$ (Panel B) and Part (ii) of Proposition 4.1 reduces the set of candidate paths from $\mathcal{J}_{\bar{m}}^{(n)}$ (Panel B) to $\mathcal{K}_{\bar{m}}^{(n)}$ (Panel C).

\bar{m} can be given by

$$\mathbb{E}(\tau) \approx e^{n\beta R_{\bar{m}j^*}} \quad (36)$$

(see Freidlin and Wentzell (1998); Beggs (2005)). Thus, the higher the individuals' opportunity cost of mistakes and greater the threshold number of deviant agents, the longer is the time it takes to escape convention \bar{m} (see also Section 7).

4. The proof of Theorem 3.1

Overall, the proof of Theorem 3.1 consists of three steps: **Step 1:** reduction using local comparison principles, **Step 2:** approximation, and **Step 3:** further reduction.

Step 1: Reduction using comparison principles

The first step in the proof of Theorem 3.1 is to show that when determining the optimal path, we can restrict ourselves to a small class of all paths consisting of straight lines parallel to the edges of the simplex. We define the set of all paths for which all transitions are from \bar{m} to some other strategy i (i.e., all paths consisting of straight lines parallel to the edges of

the simplex) (see Figure 5) as follows:

$$\mathcal{J}_{\bar{m}}^{(n)} := \{\gamma = (x_1, x_2, \dots, x_T) \in \mathcal{G}_{\bar{m}}^{(n)}; x_{t+1} = (x_t)^{\bar{m}, i} \text{ for some } i, \text{ for all } t \leq T-1\} \quad (37)$$

Further, we consider the set of paths in which all transitions are consecutive from \bar{m} to some strategy:

$$\gamma : x \xrightarrow[t_1 \text{ times}]{\text{from } \bar{m} \text{ to } i_1} y \xrightarrow[t_2 \text{ times}]{\text{from } \bar{m} \text{ to } i_2} z \cdots \xrightarrow[t_K \text{ times}]{\text{from } \bar{m} \text{ to } i_K} w$$

for some given i_1, i_2, \dots, i_K . That is, γ consists of a series of consecutive transitions first from \bar{m} to i_1 , then from \bar{m} to i_2 , ..., and finally from \bar{m} to i_K . More precisely, we write γ as

$$\gamma = (x_1, x_2, \dots, x_T) = \begin{cases} x_{t+1} = (x_t)^{\bar{m}, i_1} & \text{if } 1 \leq t \leq t_1 \\ x_{t+1} = (x_t)^{\bar{m}, i_2} & \text{if } t_1 + 1 \leq t \leq t_1 + t_2 \\ \vdots & \\ x_{t+1} = (x_t)^{\bar{m}, i_K} & \text{if } \sum_{l=1}^{K-1} t_l + 1 \leq t \leq \sum_{l=1}^K t_l =: T-1 \end{cases} \quad (38)$$

and define

$$\begin{aligned} \mathcal{K}_{\bar{m}}^{(n)} := & \{\gamma : \gamma = (x_1, x_2, \dots, x_T) \in \mathcal{J}_{\bar{m}}^{(n)}, \gamma \text{ is given by (38)} \\ & \text{for some } t_1, t_2, \dots, t_K, \text{ for some distinct } i_1, i_2, \dots, i_K\}. \end{aligned} \quad (39)$$

From Propositions 3.1 and 3.3, we obtain the following proposition. The main idea of the proofs are illustrated in Section 2.2, where we explain the argument of constructing a new lower cost path by modifying a single agent's transition using the comparison principles and comparing three paths to eliminate the intermediate path (see also Figure 5).

Proposition 4.1. *Suppose that Condition A holds. We have the following characterizations:*

(i) **Comparison principle I**

$$\min_{\gamma \in \mathcal{G}_{\bar{m}}^{(n)}} I^{(n)}(\gamma) = \min_{\gamma \in \mathcal{J}_{\bar{m}}^{(n)}} I^{(n)}(\gamma).$$

(ii) **Comparison principle II**

$$\min_{\gamma \in \mathcal{G}_{\bar{m}}^{(n)}} I^{(n)}(\gamma) = \min_{\gamma \in \mathcal{K}_{\bar{m}}^{(n)}} I^{(n)}(\gamma).$$

Proof. Part (i). In the proof, we suppress the superscript (n) . Let $\gamma = (x_1, x_2, \dots, x_T)$ be a path in $\mathcal{G}_{\bar{m}} \setminus \mathcal{J}_{\bar{m}}$. We recursively construct a new path $\tilde{\gamma} \in \mathcal{J}_{\bar{m}}$ with a cost lower than or equal to the cost of γ .

For this, let t be the greatest number such that $x_{t+1} = (x_t)^{i,l}$ with $i \neq \bar{m}, l$. We distinguish several cases. If $t = T-1$, we consider a new path $\tilde{\gamma}$ obtained by modifying the last transition as follows:

$$\tilde{\gamma} := (x_1, x_2, \dots, x_{T-1}, (x_{T-1})^{\bar{m},l}).$$

Then, we have $I(\tilde{\gamma}) = I(\gamma)$, and show that the path still exits $D(e_{\bar{m}})$. To prove this, we only need to show that if $z \notin D(e_{\bar{m}})$ then $z^{\bar{m},i} \notin D(e_{\bar{m}})$, because this implies that if $(x_{T-1})^{i,l} \notin D(e_{\bar{m}})$, then $(x_{T-1})^{\bar{m},l} \notin D(e_{\bar{m}})$. Now, suppose that $z \notin D(e_{\bar{m}})$ and that there exists k such that $\pi(\bar{m}, z) < \pi(k, z)$. Then, we have

$$[\pi(k, z^{\bar{m},i}) - \pi(\bar{m}, z^{\bar{m},i})] - [\pi(k, z) - \pi(\bar{m}, z)] = \frac{1}{n} (A_{ki} - A_{k\bar{m}} - A_{\bar{m},i} + A_{\bar{m},\bar{m}}) \geq 0$$

by Condition **A**. Thus, we have $[\pi(k, z^{\bar{m},i}) - \pi(\bar{m}, z^{\bar{m},i})] \geq [\pi(k, z) - \pi(\bar{m}, z)] > 0$ and so $z^{\bar{m},i} \notin D(e_{\bar{m}})$.

Now, suppose that $t < T-1$. Then we have $x_{t+1} = (x_t)^{i,l}$ and $x_{t+2} = (x_t)^{(i,l)(\bar{m},k)}$ for $k \neq \bar{m}$. Note that $k \neq \bar{m}$ and $l \neq i$. Now we need to distinguish four cases.

Case 1: If $k = i, l = \bar{m}$, then $x_{t+1} = (x_t)^{i,\bar{m}}, x_{t+2} = x_t$. Thus, we consider $\tilde{\gamma} = (x_1, \dots, x_t, x_{t+2}, \dots, x_T)$; clearly, $I(\tilde{\gamma}) \leq I(\gamma)$, since $c(x_t, x_{t+1}) = 0, c(x_{t+1}, x_{t+2}) \geq 0$, and $c(x_t, x_{t+2}) = 0$.

Case 2: If $k = i, l \neq \bar{m}$ then $x_{t+2} = (x_t)^{(i,l)(\bar{m},k)} = (x_t)^{\bar{m},l}$. Again, we consider the path $\tilde{\gamma} = (x_1, \dots, x_t, x_{t+2}, \dots, x_T)$ and find that $I(\tilde{\gamma}) \leq I(\gamma)$ because we have $c(x_t, x_{t+1}) = c(x_t, x_{t+2}) = \pi(m, x_t) - \pi(l, x_t)$ and $c(x_{t+1}, x_{t+2}) \geq 0$.

Case 3: If $k \neq i, l = \bar{m}$, then $x_{t+2} = x_t^{(i,\bar{m})(\bar{m},k)} = (x_t)^{i,k}$. Again, let $\tilde{\gamma} = (x_1, \dots, x_t, x_{t+2}, \dots, x_T)$. Then we have $c(x_t, x_{t+1}) = 0$ and

$$\begin{aligned} c(x_{t+1}, x_{t+2}) - c(x_t, x_{t+2}) &= c(x_t^{i,l}, x_t^{(i,l)(\bar{m},k)}) - c(x_t, x_t^{(i,k)}) \\ &= \pi(\bar{m}, x_t^{i,\bar{m}}) - \pi(k, x_t^{i,\bar{m}}) - [\pi(\bar{m}, x_t) - \pi(k, x_t)] \\ &= \frac{1}{n} (A_{\bar{m}\bar{m}} - A_{k\bar{m}} - [A_{\bar{m}i} - A_{ki}]) \geq 0 \end{aligned}$$

from **MBP**, implying that $I(\tilde{\gamma}) \leq I(\gamma)$.

Case 4: If $k \neq i, \bar{m}$ and $l \neq i, \bar{m}$, then we can apply Proposition 3.1. We modify the

path by considering the alternative transitions, $\tilde{x}_{t+1} = (x_t)^{\bar{m},l}$ and $\tilde{x}_{t+2} = (x_t)^{(\bar{m},l)(i,k)}$. If $(x_t)^{\bar{m},l} \notin D(e_{\bar{m}})$, then we define

$$\tilde{\gamma} := (x_1, x_2, \dots, x_t, (x_t)^{\bar{m},l})$$

and because $c(x_t, (x_t)^{\bar{m},l}) = c(x_t, (x_t)^{i,l})$ and $c(x_{t+1}, x_{t+2}) \geq 0$, we obtain $I(\tilde{\gamma}) \leq I(\gamma)$. If $(x_t)^{\bar{m},l} \in D(e_{\bar{m}})$, then we define

$$\tilde{\gamma} := (x_1, x_2, \dots, x_t, (x_t)^{\bar{m},l}, (x_t)^{(\bar{m},l)(i,k)}, \dots, x_T).$$

to find that $I(\tilde{\gamma}) \leq I(\gamma)$ from Proposition 3.1. Proceeding inductively we construct a path $\tilde{\gamma} \in \mathcal{J}_{\bar{m}}$ with a cost lower than or equal to the cost of γ .

Part (ii). This is the direct consequence of Proposition 3.3. Suppose that $\gamma \in \mathcal{K}_{\bar{m}}$. Then, by applying Proposition 3.3 repeatedly, we collect the same transitions and find $\tilde{\gamma} \in \mathcal{K}_{\bar{m}}$ such that $I(\tilde{\gamma}) \leq I(\gamma)$. Thus we obtain the desired result. \square

Step 2: Approximation

Proposition 4.1 shows that the cost minimizing escaping path must lie in $\mathcal{K}_{\bar{m}}^{(n)}$, which contains simple paths—namely, paths consisting of straight lines, parallel to the edges of the simplex, with t_1 transitions from \bar{m} to i_1 , t_2 transitions from \bar{m} to i_2 , and so on. Thus, we view $I^{(n)}(\gamma)$ as a function of parameters t_1, t_2, \dots, t_K . As already mentioned, the explicit expression of $I^{(n)}(\gamma)$ is somewhat complicated when n is finite; we thus study the infinite population problem

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min_{\gamma \in \mathcal{K}_{\bar{m}}^{(n)}} I^{(n)}(\gamma). \quad (40)$$

To study the problem in (40), we need to first find the continuum version of $c(x, x^{i,j})$ in (2). For this, we denote by $\bar{D}(e_i)$ the continuum version of the basin of attraction $D(e_i)$ (see Appendix A). We find that the following definition of a cost function on $\bar{D}(e_i)$ is precisely the limit of the discrete cost function at $n = \infty$ (see Lemma A.1 in Appendix A).

Definition 4.1 (Cost of a straight line path). Suppose that $p, q \in \Delta$ with $q = p + \alpha(e_i - e_j)$

for some $\alpha > 0$. If $p, q \in \bar{D}(e_{\bar{m}})$, we define

$$\bar{c}(p, q) := \frac{1}{2}(p_j - q_j)(\pi(\bar{m}, p + q) - \pi(i, p + q)). \quad (41)$$

Sometimes, we write $\bar{c}(\gamma)$ for the cost of a continuum path γ . From Lemma A.1 in Appendix A, if $x^{(n)}, y^{(n)} \in \Delta^{(n)}$ converges to p and q as $n \rightarrow \infty$ and $\gamma_{x^{(n)} \rightarrow y^{(n)}}$ is a straight line path from $x^{(n)}$ to $y^{(n)}$ in $D(e_{\bar{m}})$, then $\frac{1}{n}I^{(n)}(\gamma_{x^{(n)} \rightarrow y^{(n)}})$ converges to $\bar{c}(p, q)$. We then define a continuum analogue of the set of path $\mathcal{K}_{\bar{m}}^{(n)}$ in the limit of $n \rightarrow \infty$ and define an associated cost function.

Definition 4.2 (The set of admissible straight line paths $\mathcal{K}_{\bar{m}}$ and the related cost function). Set $\mathcal{K}_{\bar{m}}$ consists of the set of paths given by the collection of piecewise straight lines through the points

$$p^{(0)} = e_{\bar{m}}, \quad p^{(1)} = e_{\bar{m}} + t_1(e_{i_1} - e_{\bar{m}}), \dots, \quad p^{(K)} = e_{\bar{m}} + \sum_{l=1}^K t_l(e_{i_l} - e_{\bar{m}}), \quad (42)$$

where $t_l \in [0, 1]$, $p^{(l)} \in \bar{D}(e_{\bar{m}})$ for all l and $p^{(K)} \in \partial\bar{D}(e_{\bar{m}})$. We also set

$$\zeta(t) := (p^{(0)}, p^{(1)}, \dots, p^{(K)}), \quad q(t) := e_{\bar{m}} + \sum_{l=1}^K t_l(e_{i_l} - e_{\bar{m}}) = p^{(K)} \quad (43)$$

and

$$\omega(t) := \bar{c}(\zeta(t)) = \sum_{t=0}^{K-1} \bar{c}(p^{(t)}, p^{(t+1)}), \quad (44)$$

where the path $\zeta = \zeta(t) \in \mathcal{K}_{\bar{m}}$ is uniquely determined by the vector $t = (t_1, \dots, t_K)$. \square

The next result is intuitive; its (simple) proof is relegated to Appendix A.

Proposition 4.2. *We have the approximation result*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min \{ I^{(n)}(\gamma) : \gamma \in \mathcal{K}_{\bar{m}}^{(n)} \} = \min \{ \omega(t) : \zeta(t) \in \mathcal{K}_{\bar{m}} \}.$$

Proof. See Proposition A.1. \square

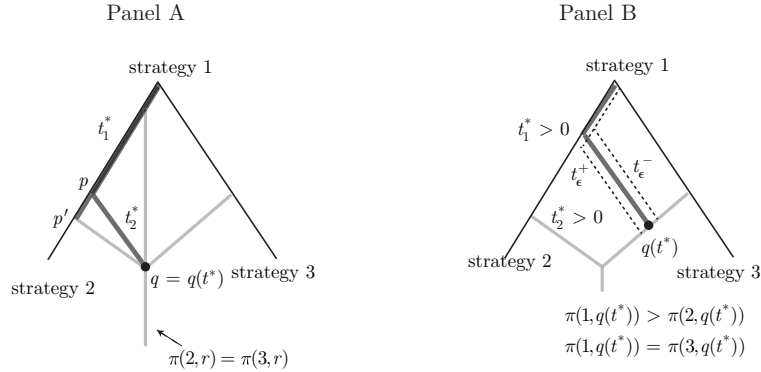


Figure 6: **Proofs of Propositions 4.3 and 4.4.** Panel A illustrates Proposition 4.3 and Panel B illustrates Proposition 4.4. Panel A illustrates two paths—one involving p and p' , and the other involving p and q . Proposition 4.3 shows that the path through p to p' has a lower cost than the path through p to q . Panel B shows that if $t_1^+ > 0$, $t_2^+ > 0$, $\pi(1, q(t^*)) > \pi(2, q(t^*))$, and $\pi(1, q(t^*)) = \pi(3, q(t^*))$, then either the path by t_ϵ^+ or the path by t_ϵ^- can be lower than or equal to the cost of the path by (t_1^*, t_2^*) .

We thus obtain the continuum version of $I^{(n)}$, $\omega(t)$, which is now the main subject of our analysis.

Step 3: Further reduction

From Proposition 4.2, we can find the minimum cost escaping path from convention \bar{m} by solving the following minimization problem:

$$\min \{ \omega(t) : \zeta(t) \in \mathcal{K}_{\bar{m}} \}. \quad (45)$$

We now explain how to further reduce the number of candidate solutions (see Panel A Figure 6). Consider a three-strategy game with the path $t^* = (t_1^*, t_2^*)$ passing through points p and $q = q(t^*)$ in Panel A of Figure 6, where $\pi(1, q(t^*)) = \pi(2, q(t^*)) = \pi(3, q(t^*))$. Consider further an alternative path directly exiting along the edge between strategies 1 and 2 at p' . To compare the costs of these two paths, using (41), we find that

$$\bar{c}(p, q) = \frac{1}{2} t_2^*(\pi(1, p+q) - \pi(3, p+q)), \quad \bar{c}(p, p') = \frac{1}{2} (p'_2 - p_2)(\pi(1, p+p') - \pi(2, p+p')).$$

Thus, from $t_2^* > p'_2 - p_2$ (see Panel A of Figure 6) and the fact that q and p' are mixed-strategy Nash equilibria,

$$\bar{c}(p, q) - \bar{c}(p, p') \geq \frac{1}{2}t_2^*[\pi(1, p) - \pi(3, p) - (\pi(1, p) - \pi(2, p))] = \frac{1}{2}t_2^*(\pi(2, p) - \pi(3, p)).$$

Clearly, $\pi(2, p) > \pi(3, p)$ from Panel A of Figure 6, because p is located at the left-hand side of the line $\pi(2, r) = \pi(3, r)$. Indeed, we can confirm that

$$\begin{aligned}\pi(2, p) - \pi(3, p) &= \pi(2, q + t_2^*(e_1 - e_3)) - \pi(3, q + t_2^*(e_1 - e_3)) \\ &= t_2^*(A_{21} - A_{23} - A_{31} + A_{33}) > 0\end{aligned}$$

by **MBP**, where we again use the fact that q is the complete mixed strategy Nash equilibrium. Here, the underlying principle is again that a path with the same consecutive transitions is cheaper than those involving different transitions.

Recall that at the end point q of the escaping path, the following constraints must be satisfied:

$$\pi(\bar{m}, q) \geq \pi(i, q) \text{ for all } i \text{ with at least one constraint binding} \quad (46)$$

The above illustration shows that at the end point of the minimal escaping path *only one* constraint is binding (i.e., $\pi(1, q) = \pi(2, q)$). The following proposition generalizes the above argument.

Proposition 4.3. *There exist t^* such that $\omega(t^*) = \min\{\omega(t) : \zeta(t) \in \mathcal{K}_{\bar{m}}\}$ and there exists k such that*

$$\pi(\bar{m}, q(t^*)) = \pi(i_k, q(t^*)) \text{ and } \pi(\bar{m}, q(t^*)) > \pi(i_l, q(t^*)) \text{ for all } l \neq k. \quad (47)$$

Proof. See Appendix C. □

From Proposition 4.3, only one binding constraint exists in (46). Proposition 4.4 further shows that only one strategy is used in the minimal escaping path. To explain the main argument for the proof, consider again the three-strategy game in Panel B of Figure 6. Suppose that the optimal solution $t^* = (t_1^*, t_2^*)$ and $t_1^*, t_2^* > 0$, $\pi(1, q(t^*)) > \pi(2, q(t^*))$ and $\pi(1, q(t^*)) = \pi(3, q(t^*))$ (see the dotted line in Panel B of Figure 6). Then, from the linearity of the payoff functions, we can find that $t_\epsilon^+ := (t_1^* + \epsilon_1, t_2^* - \epsilon_2)$ and $t_\epsilon^- := (t_1^* - \epsilon_1, t_2^* + \epsilon_2)$ which

still satisfy the constraint of escaping the basin of attraction. Then, by direct computation, we find that

$$\begin{aligned} (\omega(t_\epsilon^+) - \omega(t)) + (\omega(t_\epsilon^-) - \omega(t)) &= -H_{22}^{(1)} \epsilon_1^2 + 2H_{32}^{(1)} \epsilon_1 \epsilon_2 - H_{33}^{(1)} \epsilon_2^2 \\ &\leq -H_{22}^{(1)} \epsilon_1^2 + 2\sqrt{H_{22}^{(1)}} \sqrt{H_{33}^{(1)}} \epsilon_1 \epsilon_2 - H_{33}^{(1)} \epsilon_2^2 \leq -(\sqrt{H_{22}^{(1)}} \epsilon_1 - \sqrt{H_{33}^{(1)}} \epsilon_2)^2 < 0 \end{aligned}$$

recalling

$$H_{jk}^{(i)} := (A_{ii} - A_{ji}) - (A_{ik} - A_{jk}).$$

Thus, either $\omega(t_\epsilon^+) < \omega(t)$ or $\omega(t_\epsilon^-) < \omega(t)$ holds. The proof of Proposition 4.4 generalizes these arguments for an arbitrary n strategy game.

Proposition 4.4. *Let t^* be the solution to the minimization problem:*

$$\min\{\omega(t) : \zeta(t) \in \mathcal{K}_{\bar{m}}\}.$$

Then there exists k such that $t_k^ > 0$ and $t_l^* = 0$ for all $l \neq k$.*

Proof. See Appendix C. □

We now have all the elements in hand and Theorem 3.1 follows immediately from Propositions 4.2 and 4.4.

5. Exit problem: two-population models

This section presents our results for two-population models, which can be readily extended to multiple (more than two) population models. Consider two populations denoted by α and β , consisting of the same number of agents n , and suppose that a game is given by (A^α, A^β) , where A^κ is a $|S| \times |S|$ matrix for $\kappa = \alpha, \beta$. An α -agent playing i against j obtains a payoff A_{ij}^α , while a β -agent playing j against i obtains A_{ij}^β . We introduce the following definition.

Definition 5.1. We say that (A^α, A^β) is a coordination game if $A_{ii}^\alpha > A_{ji}^\alpha$ and $A_{ii}^\beta > A_{ij}^\beta$ for all i, j . We also say that (A^α, A^β) satisfies the *weak* marginal bandwagon property (**WBP**) if

$$A_{\bar{m}\bar{m}}^\alpha - A_{i\bar{m}}^\alpha \geq A_{\bar{m}j}^\alpha - A_{ij}^\alpha \text{ and } A_{\bar{m}\bar{m}}^\beta - A_{\bar{m}i}^\beta \geq A_{j\bar{m}}^\beta - A_{ji}^\beta \quad (48)$$

for all distinct i, j, \bar{m} .

Note that the condition in (48) is weaker than the marginal bandwagon property requiring strict inequalities. This is because we would like to apply our result to the Nash demand game (Nash (1953); see Section 7). In this section, we make the following assumption:

Condition B: Suppose that game (A^α, A^β) is a coordination game, satisfies **WBP** and admits all kinds of mixed-strategy Nash equilibria.

Suppose that $x = (x_\alpha, x_\beta) \in \square$, where $\square = \{(x_\alpha, x_\beta) : x_\alpha \in \Delta_\alpha^{(n)}, x_\beta \in \Delta_\beta^{(n)}\}$. Then the expected payoffs are similarly given by

$$\pi_\alpha(i, x) = \pi_\alpha(i, x_\beta) = \sum_{j=1}^n x_\beta(j) A_{ij}^\alpha, \quad \pi_\beta(j, x) = \pi_\beta(j, x_\alpha) = \sum_{i=1}^n x_\alpha(i) A_{ij}^\beta$$

We denote by $x^{\alpha, i, j}$ (or $x^{\beta, i, j}$) the state induced by an α -agent (or β -agent) from i to j from x . We also denote by $x^{\alpha, i, j, \eta}$ (or $x^{\beta, i, j, \eta}$) the state induced by α -agents' η -times consecutive transitions (or β -agents' η -times consecutive transitions) from i to j from x . We write e_i^α and e_j^β as the i -th and j -th standard basis for \mathbb{R}^n and thus $(e_i^\alpha, e_i^\beta) \in \square$ is a convention. We similarly define a basin of attraction of convention i as follows:

$$D(\bar{m}) := \{x = (x_\alpha, x_\beta) \in \square : \pi_\alpha(\bar{m}, x_\beta) \geq \pi_\alpha(k, x_\beta) \text{ for all } k, \pi_\beta(\bar{m}, x_\alpha) \geq \pi_\beta(k, x_\alpha) \text{ for all } k\}$$

and compute the cost functions for between states:

$$c(x, x^{\alpha, i, j}) := \pi_\alpha(\bar{m}, x) - \pi_\alpha(j, x), \quad c(x, x^{\beta, i, j}) := \pi_\beta(\bar{m}, x) - \pi_\beta(j, x) \quad (49)$$

for $x \in D(\bar{m})$. We similarly define a cost function of a path, I , as in equation (3) and the set of all paths escaping the basin of attraction of convention \bar{m} as $\mathcal{G}_{\bar{m}}^{(n)}$, as in equation (33). The following lemma is analogous to Proposition 3.1, which shows that it always costs less (or the same) to first switch from strategy \bar{m} , than from other strategies.

Lemma 5.1. *Suppose that **WBP** holds.*

$$\begin{aligned} c(x^{\beta, \bar{m}, k}, x^{(\beta, \bar{m}, k)(\alpha, j, h)}) - c(x^{\beta, i, k}, x^{(\beta, i, k)(\alpha, j, h)}) &= -A_{\bar{m}\bar{m}}^\alpha + A_{h\bar{m}}^\alpha + A_{\bar{m}i}^\alpha - A_{hi}^\alpha \leq 0 \\ c(x^{\alpha, \bar{m}, k}, x^{(\alpha, \bar{m}, k)(\beta, j, h)}) - c(x^{\alpha, i, k}, x^{(\alpha, i, k)(\beta, j, h)}) &= -A_{\bar{m}\bar{m}}^\beta + A_{\bar{m}h}^\beta + A_{i\bar{m}}^\beta - A_{ih}^\beta \leq 0. \end{aligned}$$

Proof. These are immediate from the definition. \square

Proposition 5.1 shows that Lemma 5.1 can be extended to arbitrary paths. We use Proposition 5.1 to show how to remove the transitions from $i \neq \bar{m}$ in a given path to achieve a lower cost. In Proposition 5.1, (β, i, k) , for example, refers to a transition by a β -agent from strategy i to k .

Proposition 5.1. *Suppose that **WBP** holds. We consider two paths:*

$$\begin{aligned} \gamma_1 : x &\xrightarrow{(\beta, i, k)} x^{(1)} \xrightarrow{(\alpha, j_1, k_1)} x^{(2)} \xrightarrow{(\alpha, j_2, k_2)} x^{(3)} \dots x^{(L-1)} \xrightarrow{(\alpha, j_L, k_L)} x^{(L)} \xrightarrow{(\beta, \bar{m}, l)} y \\ \gamma_2 : x &\xrightarrow{(\beta, \bar{m}, k)} y^{(1)} \xrightarrow{(\alpha, j_1, k_1)} y^{(2)} \xrightarrow{(\alpha, j_2, k_2)} y^{(3)} \dots y^{(L-1)} \xrightarrow{(\alpha, j_L, k_L)} y^{(L)} \xrightarrow{(\beta, i, l)} y \end{aligned}$$

Then, we have $I(\gamma_1) \geq I(\gamma_2)$ and a similar statement holds for a path with transitions of α agents from i to k and \bar{m} to l and transitions of α agents from \bar{m} to k and from i to l .

Proof. We find that

$$\begin{aligned} I(\gamma_1) &= c(x, x^{\beta, i, k}) + c(x^{\beta, i, k}, x^{(\beta, i, k)(\alpha, j_1, k_1)}) + c(x^{\beta, i, k}, x^{(\beta, i, k)(\alpha, j_2, k_2)}) + \dots c(x^{\beta, i, k}, x^{(\beta, i, k)(\alpha, j_L, k_L)}) \\ &\quad + c(x^{(L)}, (x^{(L)})^{(\beta, \bar{m}, l)}). \\ I(\gamma_2) &= c(x, x^{\beta, \bar{m}, k}) + c(x^{\beta, \bar{m}, k}, x^{(\beta, \bar{m}, k)(\alpha, j_1, k_1)}) + c(x^{\beta, \bar{m}, k}, x^{(\beta, \bar{m}, k)(\alpha, j_2, k_2)}) + \dots c(x^{\beta, \bar{m}, k}, x^{(\beta, \bar{m}, k)(\alpha, j_L, k_L)}) \\ &\quad + c(x^{(L)}, (x^{(L)})^{(\beta, i, l)}) \end{aligned}$$

from the fact that $c(x^{(l)}, (x^{(l)})^{\alpha, j_l, k_l}) = c(x^{\beta, i, k}, x^{(\beta, i, k)(\alpha, j_l, k_l)})$ for $l = 2, \dots, L-1$ and $c(y^{(l)}, (y^{(l)})^{\alpha, j_l, k_l}) = c(y^{\beta, \bar{m}, k}, x^{(\beta, \bar{m}, k)(\alpha, j_l, k_l)})$ for $l = 2, \dots, L-1$ (see Lemma D.1). Observe that $c(x, x^{\beta, \bar{m}, k}) = c(x, x^{\beta, i, k})$ and $c(x^{(L)}, (x^{(L)})^{(\beta, i, l)}) = c(x^{(L)}, (x^{(L)})^{(\beta, \bar{m}, l)})$. Then by applying Lemma 2 successively, we obtain the desired result. \square

We can also collect the same transitions as follows, analogously to Proposition 3.3. We also denote by $(\beta, \bar{m}, k; \eta)$ the consecutive transitions of β -agent from \bar{m} to k η -times.

Proposition 5.2. *Consider the following paths:*

$$\begin{array}{llllllllll}
\gamma & : & x & \xrightarrow{(\beta, \bar{m}, k; \eta)} & x^{\beta, \bar{m}, k; \eta} & \longrightarrow & y & \cdots & z & \xrightarrow{(\beta, \bar{m}, k; \rho)} & z^{\beta, \bar{m}, k; \rho} \\
\gamma' & : & x & \xrightarrow{(\beta, \bar{m}, k; \eta)} & x^{\beta, \bar{m}, k; \eta} & \xrightarrow{(\beta, \bar{m}, k; \rho)} & x^{(\beta, \bar{m}, k; \eta)(\beta, \bar{m}, k; \rho)} & \longrightarrow & y^{\beta, \bar{m}, k; \rho} & \cdots & z^{\beta, \bar{m}, k; \rho} \\
\gamma'' & : & x & \longrightarrow & y^{\beta, k, \bar{m}; \eta} & \cdots & z^{\beta, k, \bar{m}; \eta} & \xrightarrow{(\beta, \bar{m}, k; \eta)} & z & \xrightarrow{(\beta, \bar{m}, k; \rho)} & z^{\beta, \bar{m}, k; \rho}
\end{array}$$

where \cdots denotes the same transitions. Then either

$$I(\gamma') > I(\gamma) > I(\gamma''), I(\gamma'') > I(\gamma) > I(\gamma'), \text{ or } I(\gamma'') = I(\gamma) = I(\gamma')$$

holds. A similar statement holds for a path involving transitions of α agents' transitions.

Proof. See Appendix D. □

We also define $\mathcal{J}_{\bar{m}}^{(n)}$ and $\mathcal{K}_{\bar{m}}^{(n)}$ analogously to equations (37) and (39). That is, $\mathcal{J}_{\bar{m}}^{(n)}$ is the set of all paths in which all the transitions are from strategy \bar{m} and $\mathcal{K}_{\bar{m}}^{(n)}$ is the set of all paths consisting of consecutive transitions from \bar{m} to some other strategy. From Propositions 5.1 and 5.2, we next show that the minimum transition cost path γ involves only transitions from \bar{m} .

Proposition 5.3. *Suppose that **WBP** holds.*

(i) *We have*

$$\min\{I^{(n)}(\gamma) : \gamma \in \mathcal{G}_{\bar{m}}^{(n)}\} = \min\{I^{(n)}(\gamma) : \gamma \in \mathcal{J}_{\bar{m}}^{(n)}\}.$$

(ii) *We have*

$$\min\{I^{(n)}(\gamma) : \gamma \in \mathcal{G}_{\bar{m}}^{(n)}\} = \min\{I^{(n)}(\gamma) : \gamma \in \mathcal{K}_{\bar{m}}^{(n)}\}.$$

Proof. For the proof, we suppress the superscript (n) . Part (i). Let $\gamma \in \mathcal{G}_{\bar{m}} \setminus \mathcal{J}_{\bar{m}}$. Let the last transition of γ be from z to $z^{\beta, i, l}$ for some $i \neq \bar{m}$. Since $c(z, z^{\beta, i, l}) = c(z, z^{\beta, \bar{m}, l})$, by modifying the last transition from $z^{\beta, i, l}$ to $z^{\beta, \bar{m}, l}$ the cost will not be changed. Now, suppose that x is the last state from which a transition occurs from $i \neq \bar{m}$ in the modified path (see γ_1 in Proposition 5.1). Then, by applying Proposition 5.1, we obtain the new path whose last

transition is from $i \neq \bar{m}$ (see γ_2 in Proposition 5.1). By changing this last transition again, we can obtain a new modified path. In this way, we can remove all β -agents' transitions from $i \neq \bar{m}$. Similarly, we can also remove all α -agents' transitions from $i \neq \bar{m}$ using the corresponding part for α agents in Proposition 5.1. Thus, we can obtain the desired results. Part (ii) immediately follows from Proposition 5.2. \square

Next, we consider the continuous limit. For this, we define a cost function $\bar{c}(\mathbf{p}, \mathbf{q})$, for $\mathbf{p} = (p_\alpha, p_\beta)$, $\mathbf{q} = (q_\alpha, q_\beta) \in \square$. Let $\mathbf{q} = \mathbf{p} + \rho(e_i^\alpha - e_j^\alpha)$ or $\mathbf{q} = \mathbf{p} + \rho(e_i^\beta - e_j^\beta)$ for some $\rho > 0$. If $\mathbf{p}, \mathbf{q} \in \bar{D}(e_{\bar{m}})$,

$$\bar{c}(\mathbf{p}, \mathbf{q}) = (p_{\alpha,j} - q_{\alpha,j})(\pi_\alpha(\bar{m}, p) - \pi_\alpha(j, p)) \text{ or } \bar{c}(\mathbf{p}, \mathbf{q}) = (p_{\beta,j} - q_{\beta,j})(\pi_\beta(\bar{m}, p) - \pi_\beta(j, p)).$$

We similarly define $\mathcal{K}_{\bar{m}}$ as in Definition 4.2 and from $\zeta = \zeta(t) \in \mathcal{K}_{\bar{m}}$, where $t = (t^\alpha, t^\beta) = ((t_1^\alpha, \dots, t_K^\alpha), (t_1^\beta, \dots, t_K^\beta))$ and define $\omega(t^\alpha, t^\beta) = \sum_{s=0}^{K-1} \bar{c}(\mathbf{p}^{(s)}, \mathbf{p}^{(s+1)})$. Then, we have the following lemma.

Lemma 5.2. *Let \bar{t}^β be fixed. Then $\omega(\cdot, \bar{t}^\beta)$ is affine. A similar statement holds for the case where \bar{t}^α is fixed.*

Proof. Suppose that t_i^α is associated with α agents' transitions from \bar{m} to i . Similarly, t_j^β is associated with β agents' transitions from \bar{m} to j . Let \mathbf{p} be the state from which the transitions represented by t_i^α start. Then we find that

$$\begin{aligned} \frac{\partial \omega}{\partial t_i^\alpha} &= (\pi_\alpha(\bar{m}, p_\beta) - \pi_\alpha(i, p_\beta)) + \bar{t}_i^\beta (-A_{ii}^\beta + A_{i\bar{m}}^\beta - A_{\bar{m}\bar{m}}^\beta + A_{\bar{m}i}^\beta) \\ &\quad + \sum_{j \neq i} \bar{t}_j^\beta (-A_{ij}^\beta + A_{i\bar{m}}^\beta - A_{\bar{m}\bar{m}}^\beta + A_{\bar{m}j}^\beta) \end{aligned}$$

and observe that $\pi_\alpha(\bar{m}, p_\beta) - \pi_\alpha(i, p_\beta)$ depends only on \bar{t}^β ; this shows that $\omega(\cdot, \bar{t}^\beta)$ is affine. \square

Thus, we similarly consider

$$\min\{\omega(t) : \zeta(t) \in \mathcal{K}_{\bar{m}}\}.$$

Using the characterization that ω is affine, we show that if $t_i^{\alpha^*} > 0$ in an optimal path, then $\pi_\beta(\bar{m}, \mathbf{q}^*(t^*)) = \pi_\beta(i, \mathbf{q}^*(t^*))$ at the exit point $\mathbf{q}^*(t^*)$, where $t_i^{\alpha^*}$ denotes the transition by an α -agent from strategy \bar{m} to i .

Proposition 5.4. *Suppose that Condition **B** holds. Then, there exists $t^* \in \mathcal{K}_{\bar{m}}$ such that $\omega(t^*) = \min\{\omega(t) : \zeta(t) \in \mathcal{K}_{\bar{m}}\}$ and if $t_i^{\alpha^*} > 0$, then $\pi_\beta(\bar{m}, \mathbf{q}(t^*)) = \pi_\beta(i, \mathbf{q}(t^*))$ and if $t_j^{\beta^*} > 0$, then $\pi_\alpha(\bar{m}, \mathbf{q}(t^*)) = \pi_\alpha(j, \mathbf{q}(t^*))$, where $\mathbf{q}(t^*)$ is the end state of $\zeta(t^*)$.*

Proof. Let t^* be given such that $\omega(t^*) = \min\{\omega(t) : \zeta(t) \in \mathcal{K}_{\bar{m}}\}$. Suppose that $t_i^{\alpha^*} > 0$. The other case follows similarly. Let \bar{t}_i^α such that

$$\pi_\beta(\bar{m}, (1 - \bar{t}_i^\alpha)e_{\bar{m}}^\alpha + \bar{t}_i^\alpha e_i^\alpha) = \pi_\beta(i, (1 - \bar{t}_i^\alpha)e_{\bar{m}}^\alpha + \bar{t}_i^\alpha e_{\bar{m}}^\alpha)$$

Then, we have

$$\begin{aligned} & \pi_\beta(i, q_\alpha(t^{\alpha^*})) - \pi_\beta(\bar{m}, q_\alpha(t^{\alpha^*})) \\ &= \pi_\beta(i - m, (1 - t_i^{\alpha^*})e_{\bar{m}}^\alpha + t_i^{\alpha^*}e_i^\alpha) + \sum_{l \neq i} t_l^{\alpha^*} \pi_\beta(i - \bar{m}, e_l^\alpha - e_{\bar{m}}^\alpha) \\ &= \pi_\beta(i - m, (1 - t_i^{\alpha^*})e_{\bar{m}}^\alpha + t_i^{\alpha^*}e_i^\alpha) + \sum_{l \neq i} t_l^{\alpha^*} (A_{li}^\beta - A_{l\bar{m}}^\beta - A_{\bar{m}i}^\beta + A_{\bar{m}\bar{m}}^\beta). \end{aligned} \quad (50)$$

Now, we have two cases:

Case 1: $t_i^{\alpha^*} = \bar{t}_i^\alpha$.

Since $t^* \in \mathcal{K}_{\bar{m}}$, $\pi_\beta(i, q_\alpha(t^{\alpha^*})) - \pi_\beta(\bar{m}, q_\alpha(t^{\alpha^*})) \leq 0$, the second term in (50) ($\sum_{l \neq i} t_l^{\alpha^*} (A_{li}^\beta - A_{l\bar{m}}^\beta - A_{\bar{m}i}^\beta + A_{\bar{m}\bar{m}}^\beta)$) is non-positive. Also, **WBP** implies that the same term is non-negative, and hence zero. Thus, we have $\pi_\beta(i, q_\alpha(t^{\alpha^*})) = \pi_\beta(\bar{m}, q_\alpha(t^{\alpha^*}))$, which is the desired result.

Case 2: $0 < t_i^{\alpha^*} < \bar{t}_i^\alpha$.

Suppose that

$$\pi_\beta(\bar{m}, q_\alpha(t^{\alpha^*})) > \pi_\beta(i, q_\alpha(t^{\alpha^*})). \quad (51)$$

and

$$\pi_\beta(\bar{m}, q_\alpha(t^{\alpha^*})) = \pi_\beta(j_1, q_\alpha(t^{\alpha^*})), \pi_\beta(\bar{m}, q_\alpha(t^{\alpha^*})) = \pi_\beta(j_2, q_\alpha(t^{\alpha^*})), \dots, \pi_\beta(\bar{m}, q_\alpha(t^{\alpha^*})) = \pi_\beta(j_L, q_\alpha(t^{\alpha^*})), \quad (52)$$

where the other constraints for π_β are non-binding. We regard equations in (52) as a set of linear equations in variables, $t_{j_1}, t_{j_2}, \dots, t_{j_L}$. Then, from our assumption of the existence of

all kinds of mixed-strategy Nash equilibria, we can find functions $t_{j_1}^*(t_i), t_{j_2}^*(t_i), \dots, t_{j_L}^*(t_i)$ satisfying (51) and (52) for all $t_i \in [t_i^{\alpha*} - \epsilon, t_i^{\alpha*} + \epsilon]$ for some $\epsilon > 0$. Observe that $t_{j_1}^*(t_i), t_{j_2}^*(t_i), \dots, t_{j_L}^*(t_i)$ are affine in t_i . Then, we define $\phi(t_i) = \omega((t_i, t_{j_1}^*(t_i), t_{j_2}^*(t_i), \dots, t_{j_L}^*(t_i), \bar{t}_{i_1}, \bar{t}_{i_2}, \dots, \bar{t}_{i_{L'}}, \bar{t}^\beta))$. From Lemma 5.2, we see that $\phi(t_i)$ is affine with respect to t_i . We then find ϕ' and again have two cases.

Case 2-1. Suppose that $\phi' = 0$. Then, by increasing t_i up to $\pi_\beta(\bar{m}, \mathbf{q}(t_\alpha)) = \pi_\beta(i, \mathbf{q}(t_\alpha))$, we can find t^{**} which satisfies $\omega(t^{**}) = \omega(t^*)$ and obtain the desired properties in the proposition.
Case 2-2. Suppose that $\phi' \neq 0$. Then, we have either $\phi(t_i^{\alpha*} - \epsilon) > \phi(t_i^{\alpha*}) > \phi(t_i^{\alpha*} + \epsilon)$ or $\phi(t_i^{\alpha*} - \epsilon) < \phi(t_i^{\alpha*}) < \phi(t_i^{\alpha*} + \epsilon)$, in contradiction to the optimality of t^* . \square

Let $\mathcal{K}_{\bar{m}}^*$ be the set of all paths in $\mathcal{K}_{\bar{m}}$ that satisfy the conditions in Proposition 5.4. Then, we obviously have

$$\min\{\omega(t) : t \in \mathcal{K}_{\bar{m}}^*\} = \min\{\omega(t) : t \in \mathcal{K}_{\bar{m}}\}$$

Next, suppose that \mathbf{q}^* is the exit point of the minimum escaping path. If $\pi_\beta(\bar{m}, \mathbf{q}^*) = \pi_\beta(i, \mathbf{q}^*)$ for some i , then $\pi_\alpha(\bar{m}, \mathbf{q}^*) > \pi_\alpha(l, \mathbf{q}^*)$ for all l and vice versa. This is because if $\pi_\beta(\bar{m}, \mathbf{q}^*) = \pi_\beta(i, \mathbf{q}^*)$ and $\pi_\alpha(\bar{m}, \mathbf{q}^*) = \pi_\alpha(l, \mathbf{q}^*)$, then we can always construct the escaping path with a smaller cost by removing α -agents' (or β -agents') transitions. Thus, Proposition 5.4 implies that if $\pi_\beta(\bar{m}, \mathbf{q}^*) = \pi_\beta(i, \mathbf{q}^*)$ for some i , $t_j^{\alpha*} = 0$ for all j .

Proposition 5.5 (One-population mistakes). *Suppose that Condition **B** holds. Then there exists t^* such that $\omega(t^*) = \min\{\omega(t) : t \in \mathcal{K}_{\bar{m}}^*\}$ and t^* involves only mistakes of one population.*

Proof. Let t^* that satisfies Proposition 5.4 be given. Suppose that $t_i^{\alpha*} > 0$. The other case follows similarly. Then, by Proposition 5.4, $\pi_\beta(\bar{m}, \mathbf{q}^*) = \pi_\beta(i, \mathbf{q}^*)$ for some i . From the remarks before the proposition, we have $\pi_\alpha(\bar{m}, \mathbf{q}^*) > \pi_\alpha(l, \mathbf{q}^*)$ for all l . Again, Proposition 5.4 implies that $t_l^{\beta*} = 0$ for all l . \square

Finally, we have the following result.

Proposition 5.6. *Suppose that Condition **B** holds. Then there exists t^* such that $\min\{\omega(t) : \zeta(t) \in \mathcal{K}_{\bar{m}}^*\}$ and*

$$t^* = ((0, \dots, 0, t_l^{\alpha*}, 0, \dots, 0), t^{\beta*}) \text{ or } t^* = (t^{\alpha*}, (0, \dots, 0, t_l^{\beta*}, 0, \dots, 0)).$$

Proof. Suppose that the minimum cost escaping path involves only one population, say α -population, by Proposition 5.5. Then, $x_\beta = e_{\bar{\beta}}^{\bar{m}}$ for all x in the minimum cost escaping path. Thus we have $\pi_\alpha(i, x) = \pi_\alpha(j, x)$ for all $i, j \neq \bar{m}$ and for all x in the minimum cost escaping path. The costs of intermediate states in the minimum cost escaping path are the same; **WBP** implies that the minimum cost escaping path lies in at the boundary of the simplex, yielding the desired result. \square

Suppose that

$$R_{\bar{m}j^*} := \min\left\{(A_{\bar{m}\bar{m}}^\beta - A_{\bar{m}j}^\beta) \frac{(A_{\bar{m}\bar{m}}^\alpha - A_{j\bar{m}}^\alpha)}{(A_{\bar{m}\bar{m}}^\alpha - A_{j\bar{m}}^\alpha) + (A_{jj}^\alpha - A_{\bar{m}j}^\alpha)}, (A_{\bar{m}\bar{m}}^\alpha - A_{j\bar{m}}^\alpha) \frac{(A_{\bar{m}\bar{m}}^\beta - A_{\bar{m}j}^\beta)}{(A_{\bar{m}\bar{m}}^\beta - A_{\bar{m}j}^\beta) + (A_{jj}^\beta - A_{j\bar{m}}^\beta)}\right\}.$$

Then, we have our main result for two-population models.

Theorem 5.1. *Suppose that Condition **B** holds. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min_{\gamma \in \mathcal{G}_{\bar{m}}^{(n)}} I^{(n)}(\gamma) = R_{\bar{m}j^*}$$

where j^* satisfies

$$j^* = \arg \max\{R_{\bar{m}j} : j \neq \bar{m}\}.$$

Proof. This follows from Proposition 5.6. \square

6. Stochastic Stability

In this section, we examine the problem of finding a stochastically stable state (Foster and Young, 1990). When $\beta = \infty$, the strategy updating dynamic is called an unperturbed process, where each convention becomes an absorbing state for the dynamic. For all $\beta < \infty$, since the dynamic is irreducible, there exists a unique invariant measure. As the noise level becomes negligible ($\beta \rightarrow \infty$), the invariant measure converges to a point mass on one of the absorbing states, called a stochastically stable state. One popular way to identify a stochastically stable

state is the so-called “maxmin criterion”⁵; when some sufficient conditions are satisfied, this method, along with our results on the exit problem (Theorems 3.1 and 5.1), provides the characterization of stochastic stability. Alternatively, we can directly determine the costs of transitions from one convention to another, as explained in Section 8.

To study stochastic stability, we have to find a minimum cost path from one convention to another. More precisely, we fix conventions i and j . For one-population models, we let the set of all paths from convention i to j be

$$\mathcal{L}_{i,j}^{(n)} := \{\gamma : \gamma = (x_0, \dots, x_T) \text{ and } x_0 = e_i, x_{t+1} = (x_t)^{k,l}, \text{ for some } k, l, \text{ for all } t < T-1, \\ x_T \in D(e_j) \text{ for some } T > 0\}.$$

We can define a similar set for two-population models. We then consider the following problem:

$$C_{ij}^{(n)} := \min\{I^{(n)}(\gamma) : \gamma \in \mathcal{L}_{i,j}^{(n)}\}. \quad (53)$$

Again, when n is finite, $C_{ij}^{(n)}$ is complicated, involving many negligible terms; we thus study the stochastic stability problem at $n = \infty$, which again provides the asymptotics of the invariant measure and stochastic stability when n is large. We let

$$C_{ij} = \lim_{n \rightarrow \infty} \frac{1}{n} C_{ij}^{(n)} \quad (54)$$

and C be a $|S| \times |S|$ matrix whose elements are given by C_{ij} for $i \neq j$ (we set an arbitrary number if $i = j$). Having solved the problems in equation (53) (and (54)), the standard method to find a stochastically stable state is to construct an i -rooted tree with vertices consisting of the absorbing states and whose cost is defined as the sum of all costs between the absorbing states connected by edges. Then, the stochastic stable state is precisely the root of the minimal cost tree from among all possible rooted trees (see Young (1998b) for more details). In principle, to find a minimal cost tree (hence a stochastically stable state), we need to explicitly solve the problem in equation (53). However, in many interesting applications such as bargaining problems, the minimum cost estimates of the escaping path in Theorem 3.1 are sufficient to determine stochastic stability without knowing the true costs of transition

⁵See Young (1993, 1998b); Kandori and Rob (1998); Binmore, Samuelson, and Young (2003); Hwang, Lim, Neary, and Newton (2016)

between conventions; this method is called the “maxmin” criterion (see the papers cited in footnote 5; see also Proposition 6.1 below). More precisely, we define the incidence matrix of matrix C , $\mathbf{Inc}(C)$, as follows:

$$(\mathbf{Inc}(C))_{ij} := \begin{cases} 1 & \text{if } j = \arg \min_{l \neq i} C_{il} \\ 0 & \text{otherwise} \end{cases}$$

In words, the incidence matrix of C has 1 at the i -th and j -th position if the minimum of elements in the i th row achieves at the i -th and j -th position, and 0 otherwise. We also say that the incidence matrix of C contains a cycle, $(i, i_1, i_2, \dots, i_{t-1}, i)$, if

$$\mathbf{Inc}(C)_{ii_1} \mathbf{Inc}(C)_{i_1 i_2} \cdots \mathbf{Inc}(C)_{i_{t-1} i} > 0$$

for $t \geq 2$. Observe that we can obtain a graph by connecting the vertices of conventions i, j whose $(\mathbf{Inc}(C))_{ij}$ is 1. Also, $\mathbf{Inc}(C)$ always contains a cycle and hence the graph contains the corresponding cycle. If this cycle is unique, by removing an edge from the cycle, we can obtain a tree; this is a candidate tree to the problem of finding a minimal cost tree. Now, we are ready to state some known sufficient conditions to identify stochastic stable states.

Proposition 6.1 (Binmore, Samuelson, and Young (2003)). *Let $i^* \in \arg \max_i \min_{j \neq i} C_{ij}$.*

Suppose that either

(i) $\max_{j \neq i} C_{ji^*} < \min_{j \neq i} C_{i^* j}$

or

(ii) $\mathbf{Inc}(C)$ has a unique cycle containing i^* .

Then i^ is stochastically stable.*

Proof. See Binmore, Samuelson, and Young (2003) □

The sufficient conditions (i) and (ii) for stochastic stability in Proposition 6.1 are called the “local resistance test” and “naive minimization test,” respectively (Binmore, Samuelson, and Young, 2003). If strategy i pairwise risk-dominates strategy j (i.e., $A_{ii} - A_{ji} > A_{jj} - A_{ji}$), then under the uniform mistake model, $C_{ij} > 1/2$ and $C_{ji} < 1/2$ hold. Thus, if strategy i^* pairwise risk-dominates all strategies (called a globally pairwise risk-dominant strategy), then $C_{i^* j} > 1/2$ for all $j \neq i$ and $C_{ji^*} < 1/2$ for all $j \neq i$. Thus condition (i) in Proposition 6.1 holds and i^* is stochastically stable (see Theorem 1 in Kandori and Rob (1998) and Corollary 1 in Ellison (2000)).

The number $\min_{j \neq i} C_{ij}$ in Proposition 6.1 is, as mentioned, often called the “radius” of convention i ; this measures how difficult it is to escape from convention i (Ellison, 2000). Proposition 6.1 shows that if either (i) or (ii) holds, the state with the greatest radius (and hence the state most difficult to escape) is stochastically stable. To check whether either condition (i) or (ii) holds, clearly it is enough to know that $\min_{j \neq i} C_{ij}, \max_{j \neq i} C_{ji}$ etc.

An important consequence of our main theorem on the exit problem (Theorem 3.1) is that it provides the lower and upper bounds of the radius of convention i , $\min_{j \neq i} C_{ij}$, as follows. On the one hand, a path escaping from convention i to j (in $\mathcal{L}_{i,j}^{(n)}$) by definition exits the basin of attraction of convention i and thus $\mathcal{L}_{i,j}^{(n)} \subset \mathcal{G}_i^{(n)}$ in equation (33). Thus,

$$C_{ij}^{(n)} = \min\{I^{(n)}(\gamma) : \gamma \in \mathcal{L}_{i,j}^{(n)}\} \geq \min\{I^{(n)}(\gamma) : \gamma \in \mathcal{G}_i^{(n)}\}, \quad (55)$$

and Theorem 3.1 shows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min\{I^{(n)}(\gamma) : \gamma \in \mathcal{G}_i^{(n)}\} = \min_{j \neq i} R_{ij}. \quad (56)$$

Then equations (55) and (56) together give a lower bound for $\min_{j \neq i} C_{ij}$. On the other hand, if $\gamma_{i \rightarrow j}$ is the straight line path from convention i to j ending at the mixed strategy Nash equilibrium involving i and j , we have

$$I^{(n)}(\gamma_{i \rightarrow j}) \geq \min\{I^{(n)}(\gamma) : \gamma \in \mathcal{L}_{i,j}^{(n)}\} = C_{ij}^{(n)} \quad (57)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} I^{(n)}(\gamma_{i \rightarrow j}) = R_{ij}. \quad (58)$$

Thus, equations (57) and (58) give an upper bound for $\min_{j \neq i} C_{ij}$. These are the main contents of the following proposition.

Proposition 6.2. *Suppose Condition **A** or Condition **B** holds. Then*

- (i) $C_{ij} \leq R_{ij}$ for all i, j .
- (ii) $\min_{j \neq i} C_{ij} = \min_{j \neq i} R_{ij}$.
- (iii) $\arg \min_{j \neq i} R_{ij} \subset \arg \min_{j \neq i} C_{ij}$ for all i .

Proof. We obtain (i) by dividing equation (57) by n , taking the limit, and using (58). For (ii), from equations (55) and (56), $\lim_{n \rightarrow \infty} \frac{1}{n} C_{ij}^{(n)} \geq \min_{j \neq i} R_{ij}$, implying that $\min_{j \neq i} C_{ij} \geq$

$\min_{j \neq i} R_{ij}$. Also from (i), we have $\min_{j \neq i} C_{ij} \leq \min_{j \neq i} R_{ij}$. Thus, (ii) follows. We next prove (iii). Suppose that $j^{**} \in \arg \min_{j \neq i} R_{ij}$ and $j^* \in \arg \min_{j \neq i} C_{ij}$. Then from (i) and (ii), $R_{ij^{**}} = C_{ij^*} \leq C_{ij^{**}} \leq R_{ij^{**}}$. Thus $j^{**} \in \arg \min_{j \neq i} C_{ij}$ and we have $\arg \min_{j \neq i} R_{ij} \subset \arg \min_{j \neq i} C_{ij}$. \square

The immediate consequence of Proposition 6.2 is that $\arg \max_i \min_{j \neq i} C_{ij} = \arg \max_i \min_{j \neq i} R_{ij}$ and $\max_{j \neq i} C_{ji} \leq \max_{j \neq i} R_{ji}$. Further, if $\arg \min_{j \neq i} C_{ij}$ is unique for all i , from Proposition 6.2, the incidence matrices of C and R are the same. In general, $\arg \min_{j \neq i} C_{ij}$ may not be unique for some i . In this case, Proposition 6.2 (iii) implies that if $R_{ij} = 1$, then $C_{ij} = 1$, which, in turn, implies that whenever R yields a graph containing a unique cycle, C yields the same graph containing the unique cycle. These facts enable us to replace C in Proposition 6.1 by R —a $|S| \times |S|$ matrix consisting of R_{ij} ’s (again, we assign arbitrary numbers at the diagonal positions). This is our main result on stochastic stability.

Theorem 6.1 (Stochastic Stability). *Suppose that Condition A or Condition B holds.*

Let $i^ \in \arg \max_i \min_{j \neq i} R_{ij}$. Suppose also that either*

(i) $\max_{j \neq i} R_{ji^} < \min_{j \neq i} R_{i^*j}$*

or

(ii) $\mathbf{Inc}(R)$ has a unique cycle containing i^ .*

Then, i^ is stochastically stable.*

Proof. Let $i^* \in \arg \max_i \min_{j \neq i} R_{ij}$. From Proposition 6.2 (iii), $i^* \in \arg \max_i \min_{j \neq i} C_{ij}$. We first suppose that (i) holds. Now, Propositions 6.2 (i) and 6.2 (ii) imply that

$$\max_{j \neq i^*} C_{ji^*} \leq \max_{j \neq i^*} R_{ji^*} < \min_{j \neq i^*} R_{i^*j} = \min_{j \neq i^*} C_{i^*j}.$$

Thus, Proposition 6.1 implies that i^* is stochastically stable. Now, suppose that (ii) holds. From Proposition 6.2 (iii) and the remarks before Theorem 6.1, $\mathbf{Inc}(C)$ contains a unique cycle containing i^* , too. Thus, Proposition 6.1 again implies that i^* is stochastically stable. \square

Note that two-strategy games trivially satisfy both conditions (i) and (ii) in Theorem 6.1. Here, we can easily check that the stochastic stable state is the risk-dominant equilibrium. In particular, Kandori and Rob (1998) show that when a coordination game exhibits positive feedback (the marginal bandwagon property), a “globally pairwise risk-dominant equilibrium” is stochastically stable under the uniform mistake model (see also Binmore, Samuelson, and Young

(2003)). However, when the number of strategies exceeds two, Theorem 6.1 shows that stochastically stable states under the logit choice rule do not necessarily satisfy the criterion of pairwise risk dominance. Appendix G provides an explicit example of a four-strategy game satisfying the conditions in Theorem 6.1, for which the stochastically stable state under the logit choice rule differs from that under the uniform mistake model. To summarize, Theorem 6.1 asserts that when either condition (i) or condition (ii) is satisfied, the state with the largest radius (and hence the most difficult state to escape) is stochastically stable, in line with the existing results for uniform interaction models. However, the radius now depends on the opportunity cost of individuals' mistakes as well as the threshold number of agents inducing others to play a new best-response.

7. Application: the intentional logit solutions for the Nash demand game

In this section, we briefly discuss the application of our results to the Nash demand games (Nash, 1953). Consider two populations α and β and a bargaining set, $S \subset \mathbb{R}^2$, consisting of payoffs to two populations when they agree to split. We normalize the “disagree” point to $(0,0)$. We describe the bargaining frontier by a function f that is decreasing, differentiable and concave. We denote by \bar{s}_α and \bar{s}_β the maximum payoffs to populations α and β , respectively; that is,

$$\bar{s}_\alpha := \max\{x : (x, y) \in S\}, \quad \bar{s}_\beta := \max\{y : (x, y) \in S\}.$$

Bargaining solutions dictate how to divide the surpluses (defined by the bargaining set) between two populations⁶. We are interested in the following question: Which is the bargaining norm arising through decentralized evolutionary bargaining processes under the logit choice rule?

To answer this question, following the standard literature on evolutionary bargaining (Young, 1993, 1998a; Binmore, Samuelson, and Young, 2003; Hwang, Lim, Neary, and Newton,

⁶Three axiomatic bargaining solutions are most commonly used: the Nash bargaining solution (Nash, 1950), the Kalai-Smorodinsky bargaining solution (Kalai and Smorodinsky, 1975), and the Egalitarian bargaining solution (Kalai, 1977). See Table 7.

2016), we consider the following (discretized) Nash demand game:

$$(A_{ij}^\alpha, A_{ij}^\beta) := \begin{cases} (\delta i, f(\delta j)), & \text{if } i \leq j \\ (0, 0), & \text{if } i > j, \end{cases} \quad (59)$$

where $i \in \{1, 2, \dots, \frac{\bar{s}_\alpha - \delta}{\delta}\}$ and δ is the discretization parameter that will let $\delta \rightarrow 0$ eventually. Then, we can easily check that the game in (59) satisfies **WBP** and that Condition **B** is satisfied. Suppose that the agent’s behavioral rule follows “the intentional logit choice” defined as follows (for a more precise definition, see Appendix E). By intentional, we mean that agents choose a non-best-response strategy from the set of strategies that would give a higher payoff than the payoff at the status quo convention when adopted as a convention. Thus, the intentional logit choice rule means that agents play a non-best-response strategy among the set restricted by “intentional” behaviors with probabilities specified by the logit choice rule (see Naidu, Hwang, and Bowles (2010); Hwang, Naidu, and Bowles (2016)). Experimental evidence for intentional as well as payoff-dependent behaviors captured by the logit rule is provided in Mäs and Nax (2016) and Lim and Neary (2016).⁷ Note that for the game defined in (59), while the α -population prefers the strategy with a high i , the β -population prefers the strategy with low i . Thus, under the intentional logic dynamic, the α population chooses the suboptimal strategy from the set of strategies with higher indices than the current strategy, whereas the β population does the opposite.

Then, we find that for given m ,

⁷Lim and Neary (2016) find that individual mistakes are directed in the sense that they are group-dependent. The directed mistakes in their paper are intentional behaviors of deviant agents; for example, they find that 2.25% of subjects play mistakes when the best response is the preferred strategy, whereas 20.85% of subjects play mistakes when the best response is the less preferred strategy (Figure 5 (a) on p. 19). Also, Mäs and Nax (2016) report that the vast majority of decisions (96%) are myopic best responses, although deviations are sensitive to their costs. Particularly, they write that “deviation rates were significantly lower when subjects faced a decision where the MBR (myopic best response) was the subjects preferred option, which lends support to the assumption that deviations can be directed”(on p. 204).

	Nash Demand Game	Contract Game
Uniform	$\frac{f(s^{NB})}{s^{NB}} = -f'(s^{NB})$	$\frac{f(s^{KS})}{s^{KS}} = \frac{\bar{s}_\beta}{\bar{s}_\alpha}$
Intentional logit	$\frac{f(s^*)}{s^*} = -\frac{s^*}{f(s^*)}f'(s^*)$	$\frac{f(s^E)}{s^E} = 1$

Table 1: **Bargaining solutions**

$$\beta\text{-pop.}:\min_j \{(A_{mm}^\beta - A_{mj}^\beta) \frac{(A_{mm}^\alpha - A_{jm}^\alpha)}{(A_{mm}^\alpha - A_{jm}^\alpha) + (A_{jj}^\alpha - A_{mj}^\alpha)}\} = \min_{m>j} f(m\delta) \frac{(m-j)\delta}{m\delta} \quad (60)$$

$$\alpha\text{-pop.}:\min_j \{(A_{mm}^\alpha - A_{jm}^\alpha) \frac{(A_{mm}^\beta - A_{mj}^\beta)}{(A_{mm}^\beta - A_{mj}^\beta) + (A_{jj}^\beta - A_{jm}^\beta)}\} = \min_{m<j} m\delta \frac{f(\delta m) - f(\delta j)}{f(\delta m)} \quad (61)$$

In the second equation in (60), the minimum is taken over $\{j : j < m\}$ since the β -population prefers the strategy with lower indices; however, in the second equation in (61), the minimum is taken over $\{j : j > m\}$ since the α -population prefers the strategy with higher indices. We then find the radius for convention m :

$$\min_j R_{mj} = \min \left\{ f(m\delta) \frac{\delta}{m\delta}, m\delta \frac{f(\delta m) - f(\delta(m+1))}{f(\delta m)} \right\} \quad (62)$$

Note that the first term in the minimum of equation (62) is decreasing in m and the second term in the minimum of equation (62) is increasing in m . Thus, the maximum of (62) is achieved where the gap between the two terms in the minimum of equation (62) is smallest. More precisely, we let s^* such that

$$f(s^*) \frac{1}{s^*} = -s^* \frac{f'(s^*)}{f(s^*)}. \quad (63)$$

Then, we find

$$(\min_j R_{mj}, \arg \min_j R_{mj}) = \begin{cases} (m\delta \frac{f(\delta m) - f(\delta(m+1))}{f(\delta m)}, j+1) & \text{if } m\delta < s^* \\ (\frac{f(m\delta)}{m\delta} \delta, j-1) & \text{if } m\delta > s^* \end{cases} \quad (64)$$

Let $i^* \in \arg \max \min_j R_{ij}$. Since $\mathbf{Inc}(R)$ has a unique cycle containing i^* , Theorems 5.1

and 6.1 show that i^* is stochastically stable. Then it is straightforward to show that

$$i^* \rightarrow s^* \text{ as } \delta \rightarrow 0$$

(see Naidu, Hwang, and Bowles (2010); Hwang, Lim, Neary, and Newton (2016)); thus, the solution s^* defined in (63), is the stochastically stable bargaining norms under the intentional logit dynamic. To explain equation (63), let $(x, y) := (s^*, f(s^*))$ be the payoffs to populations α and β , respectively. Then, equation (63) can be expressed slightly more intuitively as follows:

$$y \frac{\Delta x}{x} = -x \frac{\Delta y}{y}, \quad (65)$$

where $\Delta x > 0$ if and only if $\Delta y < 0$. Under the uniform mistake model, the equation corresponding to equation (65) is given by

$$\frac{\Delta x}{x} = -\frac{\Delta y}{y}. \quad (66)$$

By rearranging (66), we easily see that equation (66) is the first-order condition maximizing xy , and hence the Nash bargaining solution satisfies equation (66). Thus, under the uniform mistake model, the Nash bargaining solution emerges as a bargaining norm through decentralized evolutionary processes. This occurs for the following two reasons. First, under the uniform mistake model, the threshold numbers of each group of population agents inducing the other population agents to play a different best response from the status quo one are the sole determinants in the evolutionary selection of bargaining norms. Second, for the Nash demand game, the threshold number for β -agents inducing new best responses of the α -agents turn out to be $\frac{\Delta x}{x}$, whereas the threshold number for α -agents inducing new best responses of the β -agents turns out to be $-\frac{\Delta y}{y}$ (recall that $\Delta x > 0$ if and only if $\Delta y < 0$).

Equation (65) shows how evolutionary selection forces are differently determined under the logit dynamics. That is, under the logit dynamics, the individual mistake costs as well as threshold numbers are important and hence the β -population's individual mistake cost y (the current share loss by deviation) and the α -population's individual mistake cost x are multiplied at the corresponding threshold values. Thus, under the intentional logit model, the bargaining norm satisfying (65) or (63) arises. Table 7 contrasts the existing bargaining solutions and the new bargaining norm obtained under the intentional logit dynamic. We leave a more detailed analysis of the bargaining solution in (64) for a future work.

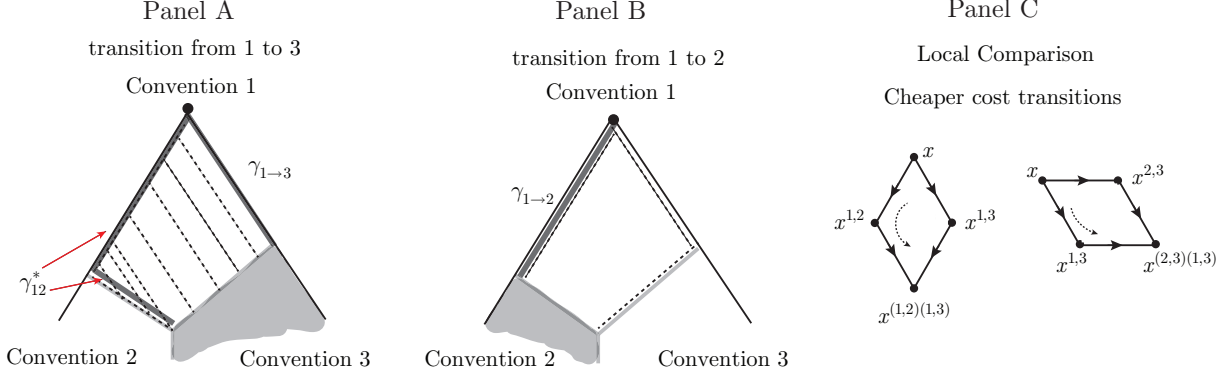


Figure 7: **Illustration of Theorem 8.1.** Panels A and B illustrate Proposition 8.1. The shaded regions show the basins of attraction of the alternative conventions. Panel C shows the comparison principles; the cost of path $x \rightarrow x^{1,2} \rightarrow x^{(1,2)(1,3)}$ is cheaper than that of path, $x \rightarrow x^{1,3} \rightarrow x^{(1,3)(1,2)}$. By adding small diamonds or parallelograms to arbitrary paths, we obtain the candidate paths. The dotted lines in Panels A and B are candidate paths for transitions from convention 1 to convention 3 and transitions from convention 1 to convention 2, respectively.

8. The direct computation of transition costs for stochastic stability

In this subsection, we briefly explain how to apply our comparison principles to directly compute the transition costs in equation (53) (see Sandholm and Staudigl (2016) for similar results from a completely different approach). For simplicity, we consider a three- strategy game, A , and without loss of generality assume that $B_{23}^{(1)} = -A_{12} + A_{21} - A_{23} + A_{32} - A_{31} + A_{13} > 0$ (otherwise, we relabel the strategies). For the three-strategy game, we are interested in estimating the minimum costs for transitions from convention 1 to convention 2 and to convention 3; that is,

$$\min_{\gamma} \{I^{(n)}(\gamma) : \gamma \in \mathcal{L}_{1,2}\}, \quad \min_{\gamma} \{I^{(n)}(\gamma) : \gamma \in \mathcal{L}_{1,3}\}.$$

To state our results, we let γ_1 be a path from convention 1 in which agents switch from strategy 1 to strategy 2 consecutively $\left\lceil n \frac{A_{11} - A_{21}}{A_{11} - A_{21} + A_{22} - A_{12}} \right\rceil - 1$ times (where $\lceil t \rceil$ is the smaller integer that exceeds t) and γ_2 be a path in the following set

$$\gamma_2 \in \{x \in D(e_1) : x^{1,2} \in D(e_2)\}$$

We let $\gamma_{12}^* := \gamma_1 \cup \gamma_2$ (see Figure 7). Note that γ_{12}^* still remains within $D(e_1)$. First, by applying the comparison principles in Propositions 3.1 and 3.3, we show that the dotted lines in Panels A and B of Figure 7 are the remaining candidate paths (see Appendix F). The idea behind this is that from $B_{23}^{(1)} > 0$ and positive feedback, we obtain the comparison principles in Panel C of Figure 7, and by adding small rectangles to the arbitrary paths, we obtain the candidate paths (the dotted lines) in Panels A and B. Panel A shows the candidate paths in the transition problem from convention 1 to convention 3 ($\mathcal{L}_{1,3}$), and Panel B similarly shows the candidate paths in the transition problem from convention 1 to convention 2 ($\mathcal{L}_{1,2}$). Now, from direct computation, we obtain the following proposition.

Proposition 8.1. *We have the following results:*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min_{\gamma} \{I^{(n)}(\gamma) : \gamma \in \mathcal{L}_{1,3}\} = \min\{\bar{c}(\gamma_{12}^*), \bar{c}(\gamma_{1 \rightarrow 3})\} \quad (67)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min_{\gamma} \{I^{(n)}(\gamma) : \gamma \in \mathcal{L}_{1,2}\} = \bar{c}(\gamma_{1 \rightarrow 2}), \quad (68)$$

where \bar{c} is the cost of the continuous path (see equation (F.10)).

Proof. See Appendix F. □

Proposition 8.1 also shows that the least cost of transition from one convention to another occurs at the boundaries of the simplex. Because of complicated geometry for games with more than three strategies, we focus only on three-strategy games; however, we believe that the problem in equation (53) for games with more than three strategies can be solved similarly through a proper modification and extension of our arguments.

9. Conclusion

Relying on positive feedback conditions and the relative strengths of these effects we developed methods to identify the most likely paths for evolutionary population dynamics under the logit rule. We analyzed the problems of exiting from a convention as well as determining the stochastically stable convention. We identified two main factors determining the cost of these paths: (1) the existence of positive feedback effects, and (2) the relative strengths of positive feedback effects. This leads us to simple but powerful comparison principles that

drastically reduce the number of candidate paths for the minimum of a suitable cost function. To sum up, we showed that the path with minimal cost involves only the repeated identical mistakes of the agents; that is, escaping path occurs along the boundary of the simplex of population states. From the solution of the exit problem, we then characterized the stochastically stable states when some sufficient and easily checkable conditions are satisfied. We also applied our finding to the bargaining problem to obtain a new bargaining norm.

The cost of the exponential better reply updating rule $c^{better}(x, x^{i,j})$ in the literature is given as follows (Sandholm, 2010; Hwang, Lim, Neary, and Newton, 2016):

$$c^{better}(x, x^{i,j}) = [\pi(i, x) - \pi(j, x)]_+ \quad (69)$$

where $[t]_+ = t$ if $t > 0$ and $[t]_+ = 0$ otherwise. The idea behind equation (69) is that the cost of playing j is now compared to the payoff of the current strategy i instead of the best-response strategy \bar{m} . This models the behavior of an agent adopting new strategy j when it gives a higher payoff than the current strategy i (consistent with the better-reply rule) and the cost of such behavior is zero. Note that if the current strategy i is the same as the best-response strategy \bar{m} , the cost of the better-reply rule in equation (69) is the same as that of the logit choice rule (2). In particular, along paths γ , γ' , and γ'' in Propositions 3.3 and 5.2, the cost of the better-reply rule is the same as that of the logit choice rule, and therefore Propositions 3.3 and 5.2 still hold for the better-reply rule in equation (69). Thus, if one shows that the minimum cost escaping paths under the better-reply rule involve only the transitions from \bar{m} , Propositions 3.3 and 5.2 can be used to estimate the minimum escaping cost.

Appendix

A. Approximation

Recall that

$$\begin{aligned} D^{(n)}(e_{\bar{m}}) &:= \{x \in \Delta^{(n)} : \pi(\bar{m}, x) \geq \pi(l, x) \text{ for all } l\} \\ \partial D^{(n)}(e_{\bar{m}}) &:= \{y \notin D^{(n)}(e_{\bar{m}}) : P_\beta(x, y) > 0 \text{ for some } x \in D^{(n)}(e_{\bar{m}})\}. \end{aligned}$$

and let

$$\bar{D}(e_{\bar{m}}) := \{p \in \Delta : \pi(\bar{m}, p) \geq \pi(l, p) \text{ for all } l\}$$

and $\partial \bar{D}(e_{\bar{m}})$ be the boundary of $\bar{D}(e_{\bar{m}})$. The following lemma serves to find the continuous version of the cost function, $c(x, x^{i,j})$.

Lemma A.1. *Let $\gamma = \gamma_{x \rightarrow y}$ be a straight-line path between x and y in $D(e_{\bar{m}}) \subset \Delta^{(n)}$ with $y = x + \frac{T}{n}(e_i - e_j)$. Then,*

$$\begin{aligned} \frac{1}{n} I^{(n)}(\gamma_{x \rightarrow y}) &= (y_i - x_i) \left[\pi\left(\bar{m}, \frac{x+y}{2}\right) - \pi\left(i, \frac{x+y}{2}\right) \right] \\ &\quad - \frac{1}{n} \frac{y_i - x_i}{2} [A_{\bar{m}i} - A_{\bar{m}j} - A_{ii} + A_{ij}]. \end{aligned} \quad (\text{A.1})$$

Proof. Since the path lies in $D(e_{\bar{m}})$ we have

$$I^{(n)}(\gamma_{x \rightarrow y}) = \sum_{t=0}^{T-1} \left[\pi\left(\bar{m}, x + \frac{t}{n}(e_i - e_j)\right) - \pi\left(i, x + \frac{t}{n}(e_i - e_j)\right) \right]. \quad (\text{A.2})$$

Now using that $1 + 2 + \dots + K - 1 = (K - 1)K/2$, we obtain

$$\sum_{t=0}^{T-1} \left(x + \frac{t}{n}(e_i - e_j)\right) = Tx + \frac{T(T-1)}{2} \frac{1}{n}(e_i - e_j) = T \frac{x+y}{2} - \frac{T}{2} \frac{1}{n}(e_i - e_j). \quad (\text{A.3})$$

By combining equations (A.2) and (A.3) and noting that $T = n(y_i - x_i)$, we obtain the desired result. \square

From (A.1), we see that the first term of the cost of a straight-line path is essentially independent of n , and by construction if $x^{(n)}, y^{(n)} \in \Delta^{(n)}$ converge to p and q as $n \rightarrow \infty$ then $\frac{1}{n}I^{(n)}(\gamma_{x^{(n)} \rightarrow y^{(n)}})$ converges to $\bar{c}(p, q)$ in equation (41).

We then have the following approximation lemma.

Lemma A.2. *Suppose that $X^{(n)} \subset \Delta^{(n)}$ and $X \subset \Delta$ and $f : X \rightarrow \mathbb{R}$ is a continuous function that admits a minimum and $f^{(n)} : X \rightarrow \mathbb{R}$. Suppose also that for all $x \in X$, there exists $\{x^{(n)}\}$ such that $x^{(n)} \in X^{(n)}$, $x^{(n)} \rightarrow x$, and $f^{(n)}(x^{(n)}) \rightarrow f(x)$. Then, we have*

$$\min_{x \in X^{(n)}} f^{(n)}(x) \rightarrow \min_{x \in X} f(x)$$

Proof. Let $\{x^{(n)}\}_n$ be the sequence of minimizers of $\min_{x \in X^{(n)}} f^{(n)}(x)$ and x^* be the minimizer of $\min_{x \in X} f(x)$. Suppose that $f^{(n)}(x^{(n)})$ does not converge to $f(x^*)$. Then there exist $\epsilon_0 > 0$ and $\{n_k\}$ such that

$$f^{(n_k)}(x^{(n_k)}) \geq f(x^*) + \epsilon_0. \quad (\text{A.4})$$

Further, from the hypothesis, we choose $y^{(n)}$ such $y^{(n)} \rightarrow x^*$. Since $\{x^{(n)}\}$ is the sequence of minimizers, we have

$$f^{(n_k)}(y^{(n_k)}) \geq f^{(n_k)}(x^{(n_k)}) \quad (\text{A.5})$$

Now, by taking $k \rightarrow \infty$ in equations (A.4) and (A.5), we find that $f(x^*) \geq f(x^*) + \epsilon_0$, which is a contradiction. \square

Thus, we obtain the following approximation result:

Proposition A.1. *We have the following result*

$$\frac{1}{n} \min\{\omega^{(n)}(t_1^{(n)}, t_2^{(n)}, \dots, t_K^{(n)}) : t^{(n)} \in \partial D^{(n)}(e_{\bar{m}})\} \rightarrow \min\{\omega(t_1, t_2, \dots, t_K) : t \in \partial \bar{D}(e_{\bar{m}})\}$$

Proof. This follows from Lemma A.2 by defining

$$X^{(n)} := \left\{ \frac{1}{n} t^{(n)} : q^{(n)}(t^{(n)}) \in \partial D^{(n)}(e_{\bar{m}}) \right\}, \quad X := \{t \in [0, 1]^K : \bar{q}(t) \in \partial \bar{D}(e_{\bar{m}})\}$$

and $f(t) := \omega(t)$ and $f^{(n)}(t) := \frac{1}{n} \omega^{(n)}(t^{(n)})$. Then clearly, $X^{(n)}, X, f^{(n)}, f$ satisfy the hypotheses of Lemma A.2. Thus we obtain the desired result. \square

B. Proof of Proposition 3.3

We denote by $I(a, b)$ the cost of path from a to b . We first need the following three lemmas.

Lemma B.1. *We have the following result.*

$$I(a, a^{\bar{m}, k, \rho}) - I(b, b^{\bar{m}, k, \rho}) = \rho[(\pi(\bar{m}, a) - \pi(k, a)) - (\pi(\bar{m}, b) - \pi(k, b))]$$

Proof. We first compute

$$\begin{aligned} I(a, a^{\bar{m}, k, \rho}) &= \pi(\bar{m}, x) - \pi(k, x) + \pi(\bar{m}, x^{\bar{m}, k}) - \pi(k, x^{\bar{m}, k}) + \dots + \pi(\bar{m}, x^{\bar{m}, k, \rho-1}) - \pi(k, x^{\bar{m}, k, \rho-1}) \\ &= \rho(\pi(\bar{m}, x) - \pi(k, x)) + \frac{\rho(\rho-1)}{2} \frac{1}{n} (-A_{\bar{m}\bar{m}} + A_{\bar{m}k} + A_{k\bar{m}} - A_{kk}) \end{aligned}$$

From this we find the desired result. □

Lemma B.2. *We have the following result.*

$$\eta[I(a, a^{\bar{m}, k, \rho}) - I(b, b^{\bar{m}, k, \rho})] + \rho[I(b^{k, \bar{m}, \eta}, b) - I(a^{k, \bar{m}, \eta}, a)] = 0$$

Proof. Using Lemma B.1 (by setting $b^{k, \bar{m}, \eta} = a$), we first find that

$$I(b^{k, \bar{m}, \eta}, b) - I(a^{k, \bar{m}, \eta}, a) = \eta[(\pi(\bar{m}, b^{k, \bar{m}, \eta}) - \pi(k, b^{k, \bar{m}, \eta})) - (\pi(\bar{m}, a^{k, \bar{m}, \eta}) - \pi(k, a^{k, \bar{m}, \eta}))].$$

Then

$$\begin{aligned} &\eta[I(a, a^{\bar{m}, k, \rho}) - I(b, b^{\bar{m}, k, \rho})] + \rho[I(b^{k, \bar{m}, \eta}, b) - I(a^{k, \bar{m}, \eta}, a)] \\ &= \eta\rho[(\pi(\bar{m}, a) - \pi(k, a)) - (\pi(\bar{m}, b) - \pi(k, b))] \\ &\quad + \eta\rho[(\pi(\bar{m}, b^{k, \bar{m}, \eta}) - \pi(k, b^{k, \bar{m}, \eta})) - (\pi(\bar{m}, a^{k, \bar{m}, \eta}) - \pi(k, a^{k, \bar{m}, \eta}))] \\ &= 0 \end{aligned}$$

□

Lemma B.3. *We have the following result.*

$$\eta[I(a^{\bar{m}, k, \rho}, b^{\bar{m}, k, \rho}) - I(a, b)] + \rho[I(a^{k, \bar{m}, \eta}, b^{k, \bar{m}, \eta}) - I(a, b)] = 0$$

Proof. Suppose that $(a, b) = (a_1, a_2, \dots, a_T)$ where $a_T = b$. Then $a_{t+1} = (a_t)^{\bar{m}, l_t}$ for some l_t . First we find

$$\begin{aligned}
& \eta[I(a_t^{\bar{m}, k, \rho}, (a_t^{\bar{m}, k, \rho})^{\bar{m}, l_t}) - I(a_t, a_t^{\bar{m}, l_t})] + \rho[I(a_t^{k, \bar{m}, \eta}, (a_t^{k, \bar{m}, \eta})^{\bar{m}, l_t}) - I(a_t, a_t^{\bar{m}, l_t})] \\
&= \eta[\pi(\bar{m}, a_t^{\bar{m}, k, \rho}) - \pi(l_t, a_t^{\bar{m}, k, \rho})] + \rho[\pi(\bar{m}, a_t^{k, \bar{m}, \eta}) - \pi(l_t, a_t^{k, \bar{m}, \eta})] \\
&= \frac{1}{n} \eta[\rho(-A_{\bar{m}\bar{m}} + A_{\bar{m}k}) - \rho(-A_{l_t\bar{m}} + A_{l_tk})] + \rho[\eta(-A_{\bar{m}k} + A_{\bar{m}\bar{m}}) - \eta(-A_{l_tk} + A_{l_t\bar{m}})] \\
&= 0
\end{aligned}$$

We thus find that

$$\begin{aligned}
& \eta[I(a^{\bar{m}, k, \rho}, b^{\bar{m}, k, \rho}) - I(a, b)] + \rho[I(a^{k, \bar{m}, \eta}, b^{k, \bar{m}, \eta}) - I(a, b)] \\
&= \sum_{t=1}^{T-1} \eta[I(a_t^{\bar{m}, k, \rho}, (a_t^{\bar{m}, k, \rho})^{\bar{m}, l_t}) - I(a_t, a_t^{\bar{m}, l_t})] + \rho[I(a_t^{k, \bar{m}, \eta}, (a_t^{k, \bar{m}, \eta})^{\bar{m}, l_t}) - I(a_t, a_t^{\bar{m}, l_t})] \\
&= 0
\end{aligned}$$

□

Proof of Proposition 3.3. We find that

$$\begin{aligned}
& \eta[I(\gamma') - I(\gamma)] + \rho[I(\gamma'') - I(\gamma)] \\
&= \underbrace{\eta[I(x^{\bar{m}, k, \eta}, x^{(\bar{m}, k, \eta)(\bar{m}, k, \rho)}) - I(z, z^{\bar{m}, k, \rho})] + \rho[I(z^{k, \bar{m}, \eta}, z) - I(x, x^{\bar{m}, k, \eta})]}_{(i)} \\
&+ \underbrace{\eta[I(x^{(\bar{m}, k, \eta)(\bar{m}, k, \rho)}, y^{\bar{m}, k, \rho}) - I(x^{\bar{m}, k, \eta}, y)] + \rho[I(x, y^{k, \bar{m}, \eta}) - I(x^{\bar{m}, k, \eta}, y)]}_{(ii)} \\
&+ \underbrace{\eta[I(y^{\bar{m}, k, \rho}, z^{\bar{m}, k, \rho}) - I(y, z)] + \rho[(I(y^{k, \bar{m}, \eta}, z^{k, \bar{m}, \eta}) - I(y, z))]}_{(iii)}
\end{aligned}$$

Then for (i), if we let $a = x^{\bar{m}, k, \eta}$ and $b = z$ in Lemma B.2, we have (i)=0. For (ii), if we let $a = x^{(\bar{m}, k, \eta)}$ and $b = y$ in Lemma B.3, we have (ii)=0. For (iii), if we let $a = y$ and $b = z$ in Lemma B.3, we have (iii)=0.

□

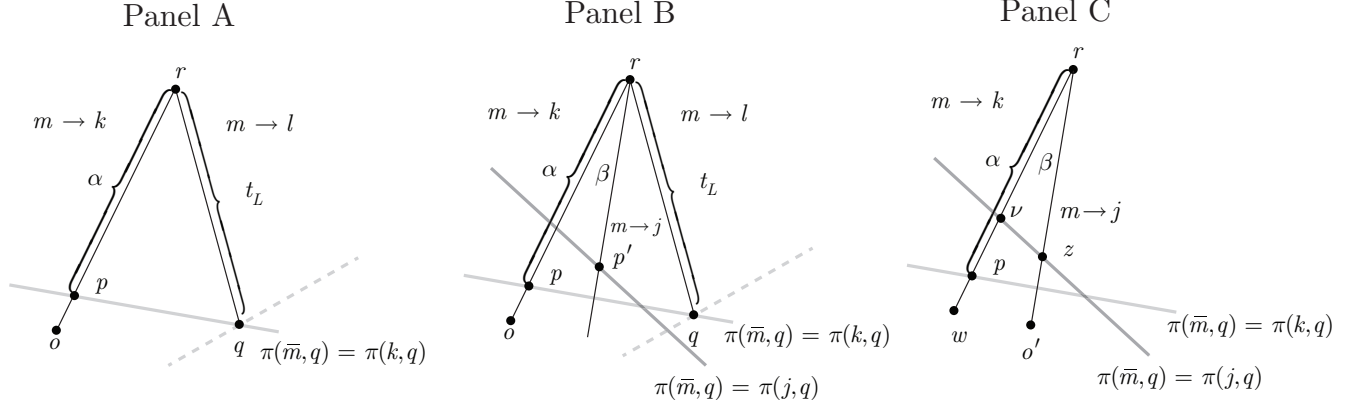


Figure C.8: Panel A shows step 1 in Lemma C.1, Panel B shows step 2 in Lemma C.1, and Panel C shows Lemma C.2

C. Proofs of Propositions 4.3 and 4.4

We will use the following notation:

$$\pi(i - j, p - q) := \pi(i, p) - \pi(j, p) - \pi(i, q) + \pi(j, q)$$

Then, **MBP** implies

$$\pi(i - j, e_i - e_k) = A_{ii} - A_{ik} - A_{ji} + A_{jk} > 0 \quad (\text{C.1})$$

and the coordination game implies that

$$\pi(i - j, e_i - e_j) = A_{ii} - A_{ij} - A_{ji} + A_{jj} > 0 \quad (\text{C.2})$$

We have the following lemma.

Lemma C.1. *Let $r \in \bar{D}(e_{\bar{m}})$ and $q \in \partial \bar{D}(e_{\bar{m}})$ and $q = r + t_L(e_l - e_{\bar{m}})$. Suppose that*

$$\pi(\bar{m}, q) = \pi(k_1, q) \text{ and } \pi(\bar{m}, q) = \pi(k_2, q).$$

Then there exists $p \in \partial D(e_{\bar{m}})$ such that $j \neq l, \bar{m}$ and $p = r + \beta(e_j - e_{\bar{m}})$, where $0 < \beta < t_L$,

$$\pi(\bar{m}, p) = \pi(j, p) \text{ and } c(r, p) < c(r, q).$$

Proof. From the condition, t_L is the length of transition from \bar{m} to l , leading to q . Choose $k := k_i \neq l$ such that

$$\pi(\bar{m}, q) = \pi(k, q).$$

Let

$$r := q + t_L(e_{\bar{m}} - e_l) \text{ and } o := r + t_L(e_k - e_{\bar{m}})$$

(i.e, r is the point obtained from q by t_L transitions from l to \bar{m} ; o can be similarly understood). Since

$$\begin{aligned} \pi(k - \bar{m}, r + t_L(e_k - e_{\bar{m}})) &= \pi(k - \bar{m}, q + t_L(e_{\bar{m}} - e_l) + t_L(e_k - e_{\bar{m}})) \\ &= t_L \pi(k - \bar{m}, e_k - e_l) > 0 \end{aligned}$$

hold from (C.1), we have

$$\pi(\bar{m}, r) \geq \pi(k, r) \text{ and } \pi(\bar{m}, o) < \pi(k, o)$$

and from the linearity of the payoff, there exists p such that $p = r + \alpha(e_k - e_{\bar{m}})$, where $\alpha > 0$ and $\pi(\bar{m}, p) = \pi(k, p)$. Then $o = p + (t_L - \alpha)(e_k - e_{\bar{m}})$. Thus

$$\begin{aligned} 0 < \pi(k, o) - \pi(\bar{m}, o) &= \pi(k - \bar{m}, p + (t_L - \alpha)(e_k - e_{\bar{m}})) \\ &\leq (t_L - \alpha) \pi(k - \bar{m}, e_k - e_{\bar{m}}) \end{aligned}$$

Thus from (C.2) we find $t_L > \alpha$ which implies that $p_k - r_k < q_l - r_l$. We divide cases.

Step 1. Suppose that $p \in \bar{D}(e_{\bar{m}})$. We also find

$$\begin{aligned} c(r, q) - c(r, p) &= \frac{1}{2} t_L \pi(\bar{m} - l, r + q) - \frac{1}{2} (p_k - r_k) \pi(\bar{m} - k, r + p) \\ &\geq \frac{1}{2} t_L (\pi(\bar{m} - l, r + q) - \pi(\bar{m} - k, r + p)) \\ &= \frac{1}{2} t_L (\pi(k, r) - \pi(l, r)) \\ &= \frac{1}{2} t_L \pi(k - l, q + t_L(e_{\bar{m}} - e_l)) \\ &= \frac{1}{2} t_L \pi(\bar{m} - l, q) + \frac{1}{2} t_L^2 \pi(k - l, e_{\bar{m}} - e_l) \\ &> 0. \end{aligned}$$

where we used $\pi(\bar{m} - l, q) \geq 0$, $\pi(k, q) = \pi(\bar{m}, q)$, and (C.1). Thus we take $\beta := \alpha$ and $j := k$ and obtain the desired result.

Step 2. Suppose that $p \notin \bar{D}(e_{\bar{m}})$. We use Lemma C.2, presented below. By taking $w = p$ and using Lemma C.2, we find z . If $z \in \bar{D}(e_{\bar{m}})$, then we set $p' = z$. Otherwise, we apply the same argument using Lemma C.2 and to find z closer to r . In this way, we can find j_1, j_2, \dots . Note that no two indices, j_1, j_2 , are the same since if $j = j_1 = j_2$ then $\pi(\bar{m} - j, r + \beta_1(e_j - e_{\bar{m}})) = \pi(\bar{m} - j_1, r + \beta_1(e_{j_1} - e_{\bar{m}})) = \pi(\bar{m} - j_2, r + \beta_2(e_{j_2} - e_{\bar{m}})) = \pi(\bar{m} - j, r + \beta_2(e_j - e_{\bar{m}}))$. Thus we find $\beta_1 = \beta_2$ which is a contradiction. Since the number of strategies is finite, we can find $z \in \bar{D}(e_{\bar{m}})$. Next, we show that $j \neq l$. If $j = l$, $\pi(\bar{m}, z) = \pi(l, z)$. Thus, we find that

$$\begin{aligned} 0 &\leq \pi(\bar{m} - l, r + t_L(e_l - e_{\bar{m}})) - \pi(\bar{m} - l, r + \beta(e_l - e_{\bar{m}})) \\ &= \pi(\bar{m} - l, (t_L - \beta)(e_l - e_{\bar{m}})) = (t_L - \beta)(-A_{\bar{m}\bar{m}} + A_{\bar{m}l} + A_{l\bar{m}} - A_{ll}) \end{aligned}$$

and thus we find $t_L \leq \beta$ which is a contradiction. So we have $j \neq l$. Then observe that $p'_j - r_j < \beta < t_L$. Then, we compute as follows:

$$\begin{aligned} c(r, q) - c(r, p') &= \frac{1}{2}t_L\pi(\bar{m} - l, r + q) - \frac{1}{2}(p'_j - r_j)\pi(\bar{m} - j, r + p') \\ &\geq \frac{1}{2}t_L(\pi(\bar{m} - l, r + q) - \pi(\bar{m} - j, r + p')) \\ &= \frac{1}{2}t_L(\pi(j, r) - \pi(l, r)) \\ &> \frac{1}{2}t_L(\pi(k, r) - \pi(l, r)) \\ &> 0. \end{aligned}$$

Thus, we can take $p = p'$.

□

Lemma C.2. Let $r \in \bar{D}(e_{\bar{m}})$. Suppose that

$$w = r + \alpha(e_k - e_{\bar{m}}), \pi(\bar{m}, w) = \pi(k, w), \text{ and } w \notin \bar{D}(e_{\bar{m}}).$$

Then there exists $j \neq k, \bar{m}$ and $\beta < \alpha$ such that

$$z := r + \beta(e_j - e_{\bar{m}}), \pi(\bar{m}, z) = \pi(j, z), \text{ and } \pi(j, r) > \pi(k, r)$$

Proof. Since $w \notin \bar{D}(e_{\bar{m}})$, there exists $j \neq k, \bar{m}$ such that $\pi(j, w) > \pi(\bar{m}, w)$. Since $\pi(\bar{m}, r) \geq \pi(j, r)$, there exists $0 < \alpha' < \alpha$ such that $\nu = r + \alpha'(e_k - e_{\bar{m}})$ and

$$\pi(\bar{m}, \nu) = \pi(j, \nu).$$

Let $o' = r + \alpha(e_j - e_{\bar{m}})$. Note that $o' = \nu - \alpha'(e_k - e_{\bar{m}}) + \alpha(e_j - e_{\bar{m}})$. Then

$$\begin{aligned} \pi(j - \bar{m}, o') &= \pi(j - \bar{m}, -\alpha'(e_k - e_{\bar{m}}) + \alpha(e_j - e_{\bar{m}})) \\ &= -\alpha'\pi(\bar{m} - j, e_{\bar{m}} - e_k) + \alpha\pi(\bar{m} - j, e_{\bar{m}} - e_j) \\ &> \alpha(\pi(j - \bar{m}, e_j - e_{\bar{m}}) - \pi(\bar{m} - j, e_{\bar{m}} - e_k)) \\ &> 0 \end{aligned}$$

Thus since $\pi(\bar{m}, r) \geq \pi(j, r)$, there exists $z = r + \beta(e_j - e_{\bar{m}})$ such that $\pi(\bar{m}, z) = \pi(j, z)$ and $\beta < \alpha$.

Next, we show that $\pi(j, r) > \pi(k, r)$. Suppose that $\pi(k, r) \geq \pi(j, r)$. We let $u(s) := r + s(e_k - e_{\bar{m}})$. Then, we have

$$\pi(\bar{m} - j, u(0)) \geq \pi(\bar{m} - k, u(0)) \tag{C.3}$$

$$\frac{d}{ds}(\pi(\bar{m} - k, u(s))) = -A_{\bar{m}\bar{m}} + A_{\bar{m}k} + A_{k\bar{m}} - A_{kk} < 0 \tag{C.4}$$

$$\frac{d}{ds}(\pi(\bar{m} - j, u(s))) = +A_{\bar{m}k} - A_{\bar{m}\bar{m}} - A_{jk} + A_{j\bar{m}} < 0 \tag{C.5}$$

$$\frac{d}{ds}(\pi(\bar{m} - k, u(s))) < \frac{d}{ds}(\pi(\bar{m} - j, u(s))) \tag{C.6}$$

Then, equations (C.3), (C.4), (C.5), and (C.6) imply that

$$0 = \pi(\bar{m} - k, w) = \pi(\bar{m} - k, u(\alpha)) < \pi(\bar{m} - j, u(\alpha)) = \pi(\bar{m} - j, r + \alpha(e_k - e_{\bar{m}})),$$

which is a contradiction to the fact that $\pi(\bar{m} - j, r + \alpha'(e_k - e_{\bar{m}})) = 0$ for $\alpha' < \alpha$. Thus, we have $\pi(j, r) > \pi(k, r)$. \square

Recall that $\mathcal{K}_{\bar{m}}$ be the set of all paths in which all transitions are from \bar{m} and the same transitions occur consecutively. Next for the convenience of readers, we restate Proposition 4.3 as follows:

Proposition C.1. *Let t^* be the solution to the minimization problem:*

$$\min\{\omega(t) : \zeta(t) \in \mathcal{K}_{\bar{m}}\}.$$

Then there exists k such that

$$\pi(\bar{m}, q(t)) = \pi(i_k, q(t)) \text{ and } \pi(\bar{m}, q(t)) > \pi(i_l, q(t)) \text{ for all } l \neq k. \quad (\text{C.7})$$

Proof. Let $t^* = (t_1, t_2, \dots, t_L)$ be the solution to the minimization problem and $(\bar{m} \rightarrow i_1, \bar{m} \rightarrow i_2, \dots, \bar{m} \rightarrow i_L)$ be the corresponding transitions. Suppose that (C.7) does not hold. Then there exists k and l where $k < l$ such that

$$\pi(\bar{m}, q(t^*)) = \pi(i_k, q(t^*)) \text{ and } \pi(\bar{m}, q(t^*)) = \pi(i_l, q(t^*))$$

We apply Lemma C.1 and can obtain a lower cost exit path, s , by removing t_L , consisting of $(\bar{m} \rightarrow i_1, \bar{m} \rightarrow i_2, \dots, \bar{m} \rightarrow i_{L-1})$ and $c(r, q) > c(r, p)$. Now applying Proposition 3.3 (or the continuous version of it), we can collect the same transitions and obtain a new path s^* in $\mathcal{K}_{\bar{m}}$ consisting of only transitions $(\bar{m} \rightarrow i_1, \bar{m} \rightarrow i_2, \dots, \bar{m} \rightarrow i_{L-1})$ such that $\omega(s^*) < \omega(t^*)$. This is a contraction to optimality of t^* . \square

Again we restate Proposition 4.4 as follows:

Proposition C.2. *Let t^* be the solution to the minimization problem:*

$$\min\{\omega(t) : \zeta(t) \in \mathcal{K}_{\bar{m}}\}.$$

Then there exists k such that $t_k^ > 0$ and $t_l^* = 0$ for all $l \neq k$.*

Proof. Let t be the solution. Then from Proposition C.1, there exists k such that

$$\pi(\bar{m}, q(t)) = \pi(i_k, q(t)) \text{ and } \pi(\bar{m}, q(t)) > \pi(i_l, q(t)) \text{ for all } l \neq k.$$

To simplify, we abuse notation by writing $k = i_k$ and $l = i_l$. Suppose that there exists l such that $t_l^* > 0$ for some $l \neq k$. Let

$$\begin{aligned} t &= (t_1^*, \dots, t_k^*, \dots, t_l^*, \dots, t_L^*) \\ t_\epsilon^+ &= (t_1^*, \dots, t_k^* + \epsilon_k, \dots, t_l^* - \epsilon_l, \dots, t_L^*) \\ t_\epsilon^- &= (t_1^*, \dots, t_k^* - \epsilon_k, \dots, t_l^* + \epsilon_l, \dots, t_L^*) \end{aligned}$$

where \dots denotes the same transitions. Then, we have

$$\begin{aligned} \pi(\bar{m}, q(t_\epsilon^+)) - \pi(i_k, q(t_\epsilon^+)) &= \pi(\bar{m}, \epsilon_k e_k - \epsilon_l e_l) - \pi(i_k, \epsilon_k e_k - \epsilon_l e_l) \\ &= \epsilon_k (A_{\bar{m}k} - A_{kk}) - \epsilon_l (A_{\bar{m}l} - A_{kl}) \end{aligned}$$

and similarly we have $\pi(\bar{m}, q(t_\epsilon^-)) - \pi(i_l, q(t_\epsilon^-)) = -\epsilon_k (A_{\bar{m}k} - A_{kk}) + \epsilon_l (A_{\bar{m}l} - A_{kl})$. Thus, we can choose small ϵ_k and ϵ_l such that

$$\begin{aligned} \pi(\bar{m}, q(t_\epsilon^+)) &= \pi(i_k, q(t_\epsilon^+)), \text{ and } \pi(\bar{m}, q(t_\epsilon^+)) > \pi(i_l, q(t_\epsilon^+)) \text{ for all } l \neq k \\ \pi(\bar{m}, q(t_\epsilon^-)) &= \pi(i_l, q(t_\epsilon^-)), \text{ and } \pi(\bar{m}, q(t_\epsilon^-)) > \pi(i_k, q(t_\epsilon^-)) \text{ for all } l \neq k, \end{aligned}$$

which show that t_ϵ^+ and t_ϵ^- both satisfy the constraints. We also find that

$$\begin{aligned} (\omega(t_\epsilon^+) - \omega(t)) - (\omega(t) - \omega(t_\epsilon^-)) &= -H_{kk}^{(\bar{m})} \epsilon_k^2 + 2H_{lk}^{(\bar{m})} \epsilon_k \epsilon_l - H_{ll}^{(\bar{m})} \epsilon_l^2 \\ &\leq -H_{kk}^{(\bar{m})} \epsilon_k^2 + 2\sqrt{H_{kk}^{(\bar{m})}} \sqrt{H_{ll}^{(\bar{m})}} \epsilon_k \epsilon_l - H_{ll}^{(\bar{m})} \epsilon_l^2 \leq -(\sqrt{H_{kk}^{(\bar{m})}} \epsilon_k - \sqrt{H_{ll}^{(\bar{m})}} \epsilon_l)^2 < 0 \end{aligned}$$

This shows that either $\omega(t_\epsilon^+) < \omega(t)$ or $\omega(t) > \omega(t_\epsilon^-)$ holds, a contradiction to the optimality of t . \square

D. Proofs of Proposition 5.2

We start with the following lemma.

Lemma D.1. *We have the following results:*

$$\begin{aligned} c(x, x^{\alpha, i, j}) &= c(z, z^{\alpha, i, j}) \text{ for all } x_\beta = z_\beta \\ c(x, x^{\beta, i, j}) &= c(z, z^{\beta, i, j}) \text{ for all } x_\alpha = z_\alpha \end{aligned}$$

Proof. This is immediate from the definition. \square

Next we show the following lemma.

Lemma D.2. *We have the following results:*

$$\eta[I(a^{\beta,\bar{m},k,\rho}, b^{\beta,\bar{m},k,\rho}) - I(a, b)] + \rho[I(a^{\beta,k,\bar{m},\eta}, b^{\beta,k,\bar{m},\eta}) - I(a, b)] = 0$$

Proof. Suppose that $(a, b) = (a_1, a_2, \dots, a_T)$ where $a_T = b$. Suppose that $a_{t+1} = (a_t)^{\beta, i_t, l_t}$. Then by applying Lemma D.1, we obtain

$$\eta[I(a_t^{\beta,\bar{m},k,\rho}, (a_t^{\beta,\bar{m},k,\rho})^{\beta, i_t, l_t}) - I(a_t, a_t^{\beta, i_t, l_t})] + \rho[I(a_t^{\beta,k,\bar{m},\eta}, (a_t^{\beta,k,\bar{m},\eta})^{\beta, i_t, l_t}) - I(a_t, a_t^{\beta, i_t, l_t})] = 0$$

We next suppose that $a_{t+1} = (a_t)^{\alpha, i_t, l_t}$.

$$\begin{aligned} & \eta[I(a_t^{\beta,\bar{m},k,\rho}, (a_t^{\beta,\bar{m},k,\rho})^{\alpha, i_t, l_t}) - I(a_t, a_t^{\alpha, i_t, l_t})] + \rho[I(a_t^{\beta,k,\bar{m},\eta}, (a_t^{\beta,k,\bar{m},\eta})^{\alpha, i_t, l_t}) - I(a_t, a_t^{\alpha, i_t, l_t})] \\ &= \eta[\pi_\alpha(\bar{m}, a_t^{\beta,\bar{m},k,\rho}) - \pi_\alpha(l_t, a_t^{\beta,\bar{m},k,\rho}) - \pi_\alpha(\bar{m}, a_t) + \pi_\alpha(l_t, a_t)] \\ & \quad + \rho[\pi_\alpha(\bar{m}, a_t^{\beta,k,\bar{m},\eta}) - \pi_\alpha(l_t, a_t^{\beta,k,\bar{m},\eta}) - \pi_\alpha(\bar{m}, a_t) + \pi_\alpha(l_t, a_t)] \\ &= \eta\rho[-A_{\bar{m}\bar{m}}^\alpha + A_{\bar{m}k}^\alpha + A_{l_t, \bar{m}}^\alpha - A_{l_t, k}^\alpha] + \rho\eta[-A_{\bar{m}k}^\alpha + A_{\bar{m}\bar{m}}^\alpha + A_{l_t, k}^\alpha - A_{l_t, \bar{m}}^\alpha] \\ &= 0 \end{aligned}$$

Thus we find

$$\begin{aligned} & \eta[I(a^{\beta,\bar{m},k,\rho}, b^{\beta,\bar{m},k,\rho}) - I(a, b)] + \rho[I(a^{\beta,k,\bar{m},\eta}, b^{\beta,k,\bar{m},\eta}) - I(a, b)] \\ &= \sum_{t=1}^{T-1} \eta[I(a_t^{\beta,\bar{m},k,\rho}, (a_t^{\beta,\bar{m},k,\rho})^{i_t, l_t}) - I(a_t, a_t^{i_t, l_t})] + \rho[I(a_t^{\beta,k,\bar{m},\eta}, (a_t^{\beta,k,\bar{m},\eta})^{i_t, l_t}) - I(a_t, a_t^{i_t, l_t})] \\ &= 0 \end{aligned}$$

\square

Lemma D.3. *We have the following results:*

- (i) $\eta[I(x^{\beta,\bar{m},k;\eta}, x^{(\beta,\bar{m},k;\eta), (\beta,\bar{m},k;\rho)}) - I(z, z^{(\beta,\bar{m},k;\rho)})] + \rho[I(z^{\beta,k,\bar{m};\eta}, z) - I(x, x^{\beta,\bar{m},k;\eta})] = 0$
- (ii) $\eta[I(x^{(\beta,\bar{m},k;\eta), (\beta,\bar{m},k;\rho)}, y^{\beta,\bar{m},k;\rho}) - I(x^{\beta,\bar{m},k;\eta}, y)] + \rho[I(x, y^{\beta,k,\bar{m};\eta}) - I(x^{\beta,\bar{m},k;\eta}, y)] = 0$
- (iii) $\eta[I(y^{\beta,\bar{m},k;\rho}, z^{\beta,\bar{m},k;\rho}) - I(y, z)] + \rho[I(y^{\beta,k,\bar{m};\eta}, z^{\beta,k,\bar{m};\eta}) - I(y, z)] = 0$

Proof. (i) By applying Lemma D.1, we find that

$$\begin{aligned}
& \eta[I(x^{\beta, \bar{m}, k; \eta}, x^{(\beta, \bar{m}, k; \eta), (\beta, \bar{m}, k; \rho)}) - I(z, z^{(\beta, \bar{m}, k; \rho)})] + \rho[I(z^{(\beta, k, \bar{m}; \eta)}, z) - I(x, x^{\beta, \bar{m}, k; \eta})] \\
&= \eta[I(x, x^{(\beta, \bar{m}, k; \rho)}) - I(z, z^{(\beta, \bar{m}, k; \rho)})] + \rho[I(z, z^{(\beta, \bar{m}, k; \eta)}) - I(x, x^{\beta, \bar{m}, k; \eta})] \\
&= \eta\rho[\pi_\beta(\bar{m}, x) - \pi_\beta(k, x) - \pi_\beta(\bar{m}, z) + \pi_\beta(k, z)] + \rho\eta[\pi_\beta(\bar{m}, z) - \pi_\beta(k, z) - \pi_\beta(\bar{m}, x) + \pi_\beta(k, x)] \\
&= 0
\end{aligned}$$

(ii) follows from by letting $a := x^{\beta, \bar{m}, k; \eta}$ and $b := y$ in Lemma D.3 and (iii) follows from by letting $a := y$ and $b := z$ in Lemma D.3. \square

Proof of Proposition 5.2. We find that

$$\begin{aligned}
& \eta[I(\gamma') - I(\gamma)] + \rho[I(\gamma'') - I(\gamma)] \\
&= \underbrace{\eta[I(x^{\beta, \bar{m}, k; \eta}, x^{(\beta, \bar{m}, k; \eta), (\beta, \bar{m}, k; \rho)}) - I(z, z^{(\beta, \bar{m}, k; \rho)})] + \rho[I(z^{(\beta, k, \bar{m}; \eta)}, z) - I(x, x^{\beta, \bar{m}, k; \eta})]}_{(i)} \\
&+ \underbrace{\eta[I(x^{(\beta, \bar{m}, k; \eta), (\beta, \bar{m}, k; \rho)}, y^{\beta, \bar{m}, k; \rho}) - I(x^{\beta, \bar{m}, k; \eta}, y)] + \rho[I(x, y^{\beta, k, \bar{m}; \eta}) - I(x^{\beta, \bar{m}, k; \eta}, y)]}_{(ii)} \\
&+ \underbrace{\eta[I(y^{\beta, \bar{m}, k; \rho}, z^{\beta, \bar{m}, k; \rho}) - I(y, z)] + \rho[I(y^{\beta, k, \bar{m}; \eta}, z^{\beta, k, \bar{m}; \eta}) - I(y, z)]}_{(iii)}
\end{aligned}$$

and Lemma D.3 (i), (ii), and (iii) show the desired result. \square

E. Intentional logit dynamics

First, we have the following logit choice rule for two populations, α and β : for $\gamma = \alpha, \beta$,

$$p_\beta^{(\gamma)}(l|x) := \frac{\exp(\beta\pi_\gamma(l, x))}{\sum_{l'} \exp(\beta\pi_\gamma(l', x))} \text{ for } \gamma = \alpha, \beta$$

and we define the set of the possible suboptimal strategies under the intentional dynamic:

$$\mathbb{S}_\gamma(x) := \{l : A_l^\gamma \geq A_{l'}^\gamma \text{ for } l' \in \arg \max_l \{\pi_\gamma(l, x)\}\}. \quad (\text{E.1})$$

The conditional probability that a $\gamma = \alpha, \beta$ agent chooses strategy l at a given state x is given by

$$p_{\beta}^{(\gamma)}(l|x) := \frac{p_{\beta}^{(\gamma)}(l|x)}{\sum_{l' \in \mathbb{S}_{\gamma}(x)} p_{\beta}^{(\gamma)}(l'|x)}$$

for $l \in \mathbb{S}_{\gamma}(x)$ and $p_{\beta}^{(\gamma)}(l|x) := 0$ otherwise.

F. Proof of Proposition 8.1

Here, we first compute the cost of the following path.

Lemma F.1. *Suppose that $x^{(n)}, y^{(n)} \in \Delta$ such that $x_i - y_i = \frac{M}{n} \frac{a}{n}$, $x_{\bar{m}} - y_{\bar{m}} = \frac{M}{n} \frac{b}{n}$ for some non-negative integers, M, a, b . Let $x^{(n)} \rightarrow p$ and $y^{(n)} \rightarrow q$ and*

$$\bar{c}(p, q) := (p_{\bar{m}} - q_{\bar{m}} + p_i - q_i) \left(\pi(\bar{m}, \frac{p+q}{2}) - \pi(j, \frac{p+q}{2}) \right). \quad (\text{F.1})$$

Then, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} I^{(n)}(\gamma_{x^{(n)} \rightarrow y^{(n)}}) = \bar{c}(p, q).$$

Proof. We let $z_{\iota_1, \iota_2} := x^{(n)} + \iota_1 \frac{a}{n} (e_i - e_j) + \iota_2 \frac{b}{n} (e_{\bar{m}} - e_j)$ and

$$\tilde{c}^{(n)}(x, y) := \frac{1}{2} (x_i - y_i) \pi(\bar{m} - j, x + y).$$

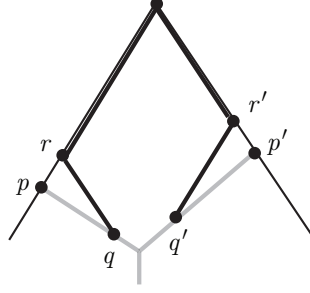


Figure F.9: Paths to be compared.

We find

$$\begin{aligned}
& \sum_{\iota=1}^M \tilde{c}^{(n)}(z_{\iota-1,\iota-1}, z_{\iota,\iota}) \\
&= \frac{1}{2} \frac{a}{n} \pi(\bar{m} - j, \frac{M+1}{2}(x^{(n)} + y^{(n)})) - \frac{1}{2} \frac{a}{n} \pi(\bar{m} - j, y^{(n)}) \\
&+ \frac{1}{2} \frac{a}{n} \pi(\bar{m} - j, \frac{M+1}{2}(x^{(n)} + y^{(n)})) - \frac{1}{2} \frac{a}{n} \pi(\bar{m} - j, x^{(n)}) - \frac{1}{2} \frac{a}{n} \pi(\bar{m} - j, \frac{b}{n} M(e_{\bar{m}} - e_j)) \\
&+ \frac{1}{2} \frac{b}{n} \pi(\bar{m} - j, \frac{M+1}{2}(x^{(n)} + y^{(n)})) - \frac{1}{2} \frac{b}{n} \pi(\bar{m} - j, x^{(n)}) - \frac{1}{2} \frac{b}{n} \pi(\bar{m} - j, \frac{b}{n} M(e_{\bar{m}} - e_j)) \\
&+ \frac{1}{2} \frac{b}{n} \pi(\bar{m} - j, \frac{M+1}{2}(x^{(n)} + y^{(n)})) - \frac{1}{2} \frac{b}{n} \pi(\bar{m} - j, x^{(n)})
\end{aligned} \tag{F.2}$$

Now since $x^{(n)} \rightarrow p$, $y^{(n)} \rightarrow q$, and $\frac{M}{n} \frac{a}{n} \rightarrow p_i - q_i$, $\frac{M}{n} \frac{b}{n} \rightarrow p_{\bar{m}} - q_{\bar{m}}$, from (F.2) we find that

$$\lim_{n \rightarrow \infty} \frac{1}{n} I^{(n)}(\gamma_{x^{(n)} \rightarrow y^{(n)}}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\iota=1}^M \tilde{c}^{(n)}(z_{\iota-1,\iota-1}, z_{\iota,\iota}) = \frac{1}{2} (p_i - q_i + p_{\bar{m}} - q_{\bar{m}}) \pi(\bar{m} - j, p + q)$$

□

We also show the following cost comparison (see Figure F). Let $\gamma_1 : r \rightarrow q$ and $\gamma_2 : r \rightarrow$

$p \rightarrow q$ where p, q, r are given by

$$p = \begin{pmatrix} 1 - p_2 \\ p_2 \\ 0 \end{pmatrix}, \quad r = \begin{pmatrix} 1 - r_2 \\ r_2 \\ 0 \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ r_2 \\ 1 - q_1 - r_2 \end{pmatrix}, \quad (\text{F.3})$$

and satisfies

$$\pi(1, p) = \pi(2, p), \quad \pi(1, q) = \pi(2, q), \quad \pi(1, r) > \pi(2, r) \quad (\text{F.4})$$

Lemma F.2. *Suppose that **MBP** holds and $-A_{12} + A_{13} + A_{21} - A_{23} - A_{31} + A_{32} > 0$. We have*

$$c(r, q) > c(r, p) + c(p, q), \quad c(r', q') < c(r', p') + c(p', q')$$

Proof. From the costs of two paths (Lemma F.1) and equation (F.4), we find that

$$\begin{aligned} c(r, q) - [c(r, p) + c(p, q)] &= \frac{1}{2}(1 - r_2 - q_1)[\pi(1, r) - \pi(3, r) - \{\pi(1, p) - \pi(3, p)\}] \\ &\quad - \frac{1}{2}(p_2 - r_2)(\pi(1, r) - \pi(2, r)) \end{aligned} \quad (\text{F.5})$$

From (F.4),

$$q_1 = \frac{A_{22} - A_{23} - A_{12} + A_{13}}{A_{11} - A_{13} - A_{21} + A_{23}} r_2 + \frac{A_{23} - A_{13}}{A_{11} - A_{13} - A_{21} + A_{23}}.$$

Thus

$$1 - r_2 - q_1 = \frac{\pi(1, r) - \pi(2, r)}{A_{11} - A_{13} - A_{21} + A_{23}} \quad (\text{F.6})$$

Further, from direct computation, we have

$$\pi(1, r) - \pi(3, r) - \{\pi(1, p) - \pi(3, p)\} = (p_2 - r_2)(A_{11} - A_{31} - (A_{12} - A_{32})) \quad (\text{F.7})$$

By substituting (F.6) and (F.7) we find that

$$c(r, q) - [c(r, p) + c(p, q)] = \frac{1}{2}(p_2 - r_2)(\pi(1, r) - \pi(2, r)) \frac{-A_{12} + A_{13} + A_{21} - A_{23} - A_{31} + A_{32}}{A_{11} - A_{21} - (A_{13} - A_{23})}$$

which gives the first comparison result. The second inequality follows by interchanging the roles of strategies 2 and 3. \square

Now we give the sketch of the proof for Proposition 8.1, which we state once again below for the convenience of readers.

Proposition F.1. *We have the following results:*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min_{\gamma} \{I^{(n)}(\gamma) : \gamma \in \mathcal{L}_{1,3}\} = \min\{\bar{c}(\gamma_{12}^*), \bar{c}(\gamma_{1 \rightarrow 3})\} \quad (\text{F.8})$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min_{\gamma} \{I^{(n)}(\gamma) : \gamma \in \mathcal{L}_{1,2}\} = \bar{c}(\gamma_{1 \rightarrow 2}), \quad (\text{F.9})$$

where

$$\bar{c}(\gamma_{12}^*) = \frac{1}{2}(p_1 - q_1 + p_2 - q_2)(\pi(1, p) - \pi(3, p)) \quad (\text{F.10})$$

and p is the mixed strategy Nash equilibrium and q is the complete mixed strategy Nash equilibrium.

Sketch of proof. Here we provide the sketch of the proof. Let

$$\mathcal{E} := \{x \in D(e_1) : x^{1,2} \in D(e_2)\}.$$

We first prove (F.8). The proof consists of two parts. Let γ^* be such that $c(\gamma^*) = \min_{\gamma} \{c(\gamma) : \gamma \in \mathcal{L}_{1,3}\}$. First suppose that $\gamma^* \in D(e_1)$. We show that for all $x \in \gamma^*$, (i) $x^{3,2} \notin \gamma^*$; (ii) if $x^{2,3} \in \gamma^*$, then $x \in \mathcal{E}$; and (iii) $x \rightarrow x^{1,3} \rightarrow x^{(1,3)(1,2)} \notin \gamma^*$ (i.e., transitions from x through $x^{1,3}$ to $x^{(1,3)(1,2)}$). For (i) and (ii), let t be the greatest number such that $x_{t+1} = (x_t)^{i,l}$ and $i \neq 1$. If $t = T - 1$, then it must be the case that $i = 2$ and $l = 3$ and $x \in \mathcal{E}$. Then choose again $t < T - 1$ be the greatest number such that $x_{t+1} = (x_t)^{i,l}$ and $i \neq 1$. If $i = 3$, we can find a lower cost path by using Proposition 3.1, which is a contradiction. If $i = 2$ and $x \notin \mathcal{E}$, we similarly find a lower cost path, using Proposition 3.1, which again is a contradiction. Thus we prove (i) and (ii). Part (iii) follows from Lemma 3.2. Thus, the paths satisfying conditions (i), (ii), and (iii) are candidate solution paths (see Panel A, Figure 7). We next obtain two paths in (F.9) by using the first part of Lemma F.2. Finally we show that γ^* cannot lie in the interior of $D(e_2)$. To do this, we compare the following paths

$$\gamma_2 : x \rightarrow x^{1,3}, \quad \gamma_1 : x \rightarrow x^{2,3} \rightarrow x^{(2,3)(1,2)}$$

in $D(e_2)$ and find that $c(\gamma_2) > c(\gamma_1) = -A_{21} + A_{22} > 0$. We also compare

$$\gamma_2 : x \rightarrow x^{1,2} \rightarrow x^{(1,2)(2,3)}, \quad \gamma_1 : x \rightarrow x^{1,3}$$

in $D(e_2)$ and find that $c(\gamma_2) > c(\gamma_1) = A_{22} - A_{23} > 0$. From these two comparisons, we can easily show that any path lying in the interior of $D(e_2)$ cannot be the minimum cost path. Further, (F.9) follows from the second part of Lemma F.2 and similar arguments. \square

G. An example: exit and stochastic stability problems of four strategy games

Consider a symmetric game, A , given by

$$A = \begin{pmatrix} 16 & 4 & 8 & 2 \\ 6 & 16 & 8 & -2 \\ 4 & -2 & 19 & 2 \\ 2 & 4 & 6 & 13 \end{pmatrix}$$

Then, it can be checked whether game A satisfies Conditions A. In this example, we find that

$$R = \begin{pmatrix} * & \frac{25}{11} & \frac{72}{23} & \frac{98}{25} \\ \frac{36}{11} & * & \frac{162}{29} & \frac{8}{3} \\ \frac{121}{46} & \frac{121}{58} & * & \frac{169}{48} \\ \frac{121}{50} & \frac{25}{6} & \frac{121}{48} & * \end{pmatrix}, \quad \mathbf{Inc}(R) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Thus, $\mathbf{Inc}(R)$ contains a cycle $(1, 2, 4, 1)$ and

$$\min_j R_{1j} = \frac{25}{11}, \quad \min_j R_{2j} = \frac{8}{3}, \quad \min_j R_{3j} = \frac{121}{58}, \quad \min_j R_{4j} = \frac{121}{50}$$

and

$$2 = \arg \max_i \min_j R_{ij}$$

which is contained in the cycle $(1, 2, 4, 1)$. Thus, condition (ii) in Theorem 6.1 is satisfied and convention 2 is stochastically stable. We can check that the stochastically stable state under the uniform mistake model is different from convention 2. In fact, we find R^U (matrix

R under the uniform mistake model) and $\mathbf{Inc}(R^U)$ as follows:

$$R^U = \begin{pmatrix} * & \frac{5}{11} & \frac{12}{23} & \frac{14}{25} \\ \frac{6}{11} & * & \frac{18}{29} & \frac{4}{9} \\ \frac{11}{23} & \frac{11}{29} & * & \frac{13}{24} \\ \frac{11}{25} & \frac{5}{9} & \frac{11}{24} & * \end{pmatrix}, \quad \mathbf{Inc}(R^U) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Thus, $\mathbf{Inc}(R^U)$ again contains a cycle $(1, 2, 4, 1)$; however, we have

$$1 = \arg \max_i \min_j R_{ij}^U$$

Thus, convention 1 is stochastically stable and this example shows that, in general, the prediction of a long-run equilibrium under the logit choice model is different from the one under a uniform mistake model.

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